Rayleigh limit of generalized Lorenz-Mie theory for on-axis beams and its relationship with the dipole theory of forces. Part I: Non dark axisymmetric beams of the first kind, with the example of Gaussian beams

Gérard Gouesbet[1] and Leonardo André Ambrosio[2]

1. CORIA-UMR 6614- Normandie Université
CNRS-Université et INSA de Rouen
Campus Universitaire du Madrillet
76800, Saint-Etienne-du Rouvray, France.
2.Department of Electrical and Computer Engineering
São Carlos School of Engineering, University of São Paulo
400 Trabalhador são-carlense Ave., São Paulo, SP 13566-590, Brazil.
Corresponding author: gouesbet@coria.fr

January 25, 2021

Abstract

The Rayleigh limit of the generalized Lorenz-Mie theory (GLMT) has been recently examined in the case of off-axis circularly symmetric Bessel beams, thereafter in the case of on-axis circularly symmetric Bessel beams, the on-axis case providing an easier framework for the understanding of the optical forces exerted in the Rayleigh limit of GLMT. This work is here extended to the case of non dark on-axis axisymmetric beams of the first kind This encompasses the case of zeroth-order circularly symmetric Bessel beams and the case of localized models of Gaussian beams. Three kinds of optical forces are exhibited, namely traditional gradient and scattering forces, plus another kind of forces which is for the time being denoted as non standard forces. The relationship between the Rayleigh limit of GLMT and the dipole theory of forces is discussed, to the best of our present understanding.

Keywords: optical forces; gradient and scattering forces; Rayleigh regime; generalized Lorenz-Mie theory; dipole theory of forces.

1 Introduction.

The generalized Lorenz-Mie theory *stricto sensu* (GLMT) is a rigorous analytical theory describing the interaction between an arbitrary electromagnetic shaped beam (or structured beam) and a homogeneous spherical particle described by its diameter d and its complex index of refraction n_p (supposed to be real in the context of the present paper), e.g. [1], [2], [3]. This theory had many applications, particularly in the field of optical particle characterization, e.g. [4], [5]. Another topic concerns the mechanical effects of light (radiation pressure forces and torques) which have been investigated as well by using GLMT, including the prediction of reverse radiation forces, e.g. [6], [7], [8], [9] with the case of spheroids considered in the framework of an extended GLMT [10], [11]. A review devoted to GLMTs (in a plural extended meaning) and mechanical effects of light is available from [12] with about 300 references.

However, strange as it may be, the application of GLMT to the case of small particles in the Rayleigh regime (point-like particles) has not been considered in a systematic way until recently, namely until [13] which dealt with longitudinal optical forces exerted by off-axis circularly symmetric Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. One of the interests of the Rayleigh regime with respect to the case of large particles is that it allows one to manipulate more easily the formal computations and therefore to provide a better picture of the physical mechanisms at work.

In particular, the GLMT may describe the incident fields using expansions either in terms of scalar potentials [1] or in terms of vector spherical wave functions [14] and then encodes the structure of the beam in a set of beam shape coefficients (BSCs) usually denoted as $g_{n,TM}^{m}$ and $g_{n,TE}^{m}$ (n from 1 to ∞, m from $(-n)$ to $(+n)$, TM for "Transverse Magnetic" and TE for "Transverse Electric"). As a result the Rayleigh limit of the GLMT expresses optical forces in terms of BSCs and will then reveal intimate features of these optical forces which are not revealed when we use the more traditional dipole theory of forces.

In [13], the example of circularly symmetric Bessel beams has been chosen because (i) for such beams, the gradient with respect to the axis of propagation z of $|\mathbf{E}|^2$ is zero and, therefore, gradient forces were expected to be zero as well (an expectation which has been confirmed), allowing one to concentrate on scattering forces and (ii) the BSCs of such beams were known under closed forms, allowing an easier implementation of the formal computations [15], [16], [17], see also [18], [19], [20]. The complication induced by the fact that, for the sake of generality, an off-axis configuration had been considered, was supposed to be in part compensated by the fact the analysis was supposed to ignore, at least partly, the details involved by the existence of gradient forces if they had not been equal to zero.

The main findings of [13] are as follows. While the dipole theory of forces expresses optical forces using the total electric field $\mathbf{E}$, i.e. using all partial waves of all orders (from $n = 1$ to ∞), the Rayleigh limit of the GLMT only uses BSCs associated with $n = 1$ and $n = 2$. Due to this fact, as it stood at this moment, the equivalence between the Rayleigh limit of GLMT and the traditional dipole theory of forces was questionable. Furthermore, beside the traditional gradient forces (although zero in the case of circularly symmetric Bessel beams) proportional to α^3 (in which $\alpha = \pi d/\lambda$, with λ the wavelength) and to the gradient of $|\mathbf{E}|^2$, and the traditional scattering forces proportional to α^6 and to the Poynting vector, another kind of forces proportional to α^6 but *not* proportional to the Poynting vector was observed. These forces had no counterpart in a discussion of longitudinal optical forces exerted by Gaussian beams in the weak confinement limit in the Rayleigh limit using GLMT, such as reported by Lock in [21], which exhibited only traditional gradient and scattering forces of the traditional dipole theory of forces. The "new" kind of forces described in [13] depends on the axicon angle which intervenes in the mathematical description of Bessel beam and they therefore have been called axicon forces. The fact that such forces were not involved in the dipole theory of forces as expounded in [21] and in other papers reinforced the significance of the question to know whether the Rayleigh limit of GLMT is identical or not, and in which way, to the dipole theory of forces. Although not completely solved in utmost rigor, this issue will be discussed at the end of the present paper.

After [13] devoted to longitudinal forces, another paper was devoted to transverse forces exerted by circularly symmetric Bessel beams on Rayleigh particles [22]. This paper revealed the existence of transverse axicon forces similar to the longitudinal axicon forces exhibited in [13]. Furthermore, in both cases, these extra-forces are zero when the axicon angle is zero (or become a traditional scattering force in this limit). These facts were a motivation to give the name of axicon forces to the extra-forces thus involved. Also, it was a clue that they might be specific of all kinds of beams exhibiting an axicon angle and that, for some reason, still to be revealed, the dipole theory of forces was lacking something in the case of beams exhibiting an axicon angle, making the identification between the Rayleigh limit of the GLMT and the dipole theory of forces more questionable.

To advance toward the solution to such issues, it has been decided to simplify the formalism at hand by considering circularly symmetric Bessel beams in the case of an on-axis configuration rather than in a case of an off-axis configuration [23]. The present paper reports on another step, still dealing with an on-axis configuration, particularly with longitudinal forces insofar as the transverse forces will be found to be zero. Rather than considering specifically circularly symmetric Bessel beams, it deals with a large class of beams which encompasses circularly symmetric Bessel beams of order $l = 0$ (no topological charge) and Gaussian beams. Revisiting Gaussian beams and relaxing the assumption of a weak confinement limit used by Lock [21], we shall observe that extra-forces exist as well in the case of Gaussian beams although they do not possess any axicon angle. Therefore, rather than using the terminology "axicon forces", we have used the less specific terminology "non-standard forces". Whether we can say more on the nature of these non-standard forces is another subject of the present paper.

The class of beams to be considered is formed of (on-axis) non dark axisymmetric beams of the first kind. Before embarking in the bulk of the present paper, the terminology used in the title requires to be explained. In the present paper, axisymmetric beams are defined as beams which, when propagating along the z-direction, possess a component S_z of the Poynting vector which does not depend on the azimuthal angle φ [24], [25]. Circularly symmetric beams [15], [16], [17], may then be defined as axisymmetric beams which possess a supplementary symmetry property, namely that the transverse component $S_t = \sqrt{S_x^2 + S_y^2}$ of the Poynting vector as well does not depend on φ (although S_x and S_y may individually depend on this angle). BSCs of axisymmetric beams satisfy very appealing expressions when all the BSCs $g_{n,X}^{\pm 1}$ ($X = TM, TE$) are different from 0, e.g. Eqs.(66) and (78) and a remark between Eqs.(65) and (66) in [25], namely:

$$\left.\begin{array}{c} g_{n,TM}^{m} = g_{n,TE}^{m} = 0,\, m \neq \pm 1 \\ g_n/2 = g_{n,TM}^{1} = g_{n,TM}^{-1}/K = -i\varepsilon g_{n,TE}^{1} = i\varepsilon g_{n,TE}^{-1}/K \end{array}\right\} \tag{1}$$

in which K describes the polarization state of the beam and $\varepsilon = \pm 1$ defines the beam-propagation direction. Eq.1 shows that all bi-index BSCs $g_{n,X}^{\pm 1}$ can be expressed in terms of uni-index BSCs g_n, leading to very significant simplifications of the formalism. Such axisymmetric beams are called axisymmetric beams of the first kind and are considered in the present paper. It may happen that not all $g_{n,X}^{\pm 1}$ are different from 0. This is the case for on-axis circularly symmetric Bessel beams when the order is equal to $l = (+2)$, in which case the BSCs $g_{n,X}^{-1}$ are 0, or $l = (-2)$ in which case the BSCs $g_{n,X}^{1}$ are 0, e.g. [23].

Such beams are from now on called axisymmetric of the second kind. Furthermore, we may consider whether beams are dark or non dark. By definition, an on-axis axisymmetric beam is said to be dark (more explicitly dark along its axis of symmetry) iff (if and only if) the $z-$component of its Poynting vector S_z taken along the z-direction on its axis (i.e. explicitly at $\theta = 0$) is zero [25]. A beam which is non dark is called a non dark beam (or a bright beam). We furthermore possess a darkness theorem telling us that, if all the BSCs $g_{n,X}^{\pm 1}$ are zero, then the beam is dark (on the axis), e.g. [26]. This implies that axisymmetric beams of the first and of the second kinds discussed above are non dark. There are therefore three kinds of axisymmetric beams to be considered (i) axisymmetric beams of the first kind (which are non dark), (ii) axisymmetric beams of the second kind (which are non dark) and (iii) axisymmetric dark beams. The present paper is devoted to item (i) while items (ii) and (iii) are postponed to future works. Examples of (non dark) axisymmetric beams of the first kind are plane waves, on-axis spherical wave fronts, circularly symmetric Bessel beams of order $l = 0$, and Gaussian beams [23], [26].

The paper is organized as follows. Section 2 deals with the expressions of optical forces. Section 3 deals with the expressions for the Poynting vector components. Section 4 develops the example of Gaussian beams. In section 5, optical forces observed in Section 4 are interpreted both in the framework of the

Rayleigh limit of GLMT and in the framework of the traditional dipole theory of forces. Section 6 is a conclusion. Main equations are Eqs.8, 15, 17, 18 for optical forces, and Eqs.44, 58, 71, 73 for the Poynting vector.

2 Optical forces in the Rayleigh limit of GLMT.

2.1 General expressions.

For reader convenience, the general expressions concerning the radiation pressure cross-sections $C_{pr,i}$ ($i = x, y, z$) are recalled in this subsection, e.g. [27], with a normalization condition reading as $E_0 H_0^*/2 = 1$ (E_0 and H_0 being electric and magnetic field strengths respectively, with the star denoting a complex conjugation). Optical forces F_i are then related to the radiation pressure cross-sections $C_{pr,i}$ by $F_i = C_{pr,i}/c$ in which c is the speed of light, e.g. [28], p.34. The time-dependence of the wave is taken to be $\exp(i\omega t)$ which is the usual convention in GLMT.

The transverse cross-section $C_{pr,x}$ reads as:

$$\begin{aligned} C_{pr,x} &= \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1\neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \\ &\times [\mathrm{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p})(\frac{\delta_{m,n+1}}{m^2} - \frac{\delta_{n,m+1}}{n^2}) \\ &+ \frac{2n+1}{n^2(n+1)^2} \delta_{nm} \, \mathrm{Re}(T_{mn}^{p-1} - T_{nm}^{-p} - 2V_{mn}^{p-1} + 2V_{nm}^{-p})] \end{aligned} \quad (2)$$

in which:

$$S_{nm}^p = (a_n + a_m^*) g_{n,TM}^p g_{m,TM}^{p+1*} + (b_n + b_m^*) g_{n,TE}^p g_{m,TE}^{p+1*} \quad (3)$$

$$T_{nm}^p = -i(a_n + b_m^*) g_{n,TM}^p g_{m,TE}^{p+1*} + i(b_n + a_m^*) g_{n,TE}^p g_{m,TM}^{p+1*} \quad (4)$$

$$U_{nm}^p = a_n a_m^* g_{n,TM}^p g_{m,TM}^{p+1*} + b_n b_m^* g_{n,TE}^p g_{m,TE}^{p+1*} \quad (5)$$

$$V_{nm}^p = i b_n a_m^* g_{n,TE}^p g_{m,TM}^{p+1*} - i a_n b_m^* g_{n,TM}^p g_{m,TE}^{p+1*} \quad (6)$$

in which the notations are the ones of [1]. In particular, a_n and b_n are the usual Mie coefficients of the usual Lorenz-Mie theory, λ is the wave-length, and δ_{nm} is the Kronecker symbol. The y-component $C_{pr,y}$ is obtained from the $x-$component by changing Re to Im, while the longitudinal component reads as:

$$
\begin{aligned}
C_{pr,z} \quad = \quad & \frac{\lambda^2}{\pi}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\{\frac{1}{(n+1)^2}\frac{(n+1+|m|)!}{(n-|m|)!} \\
& \text{Re}[(a_n+a_{n+1}^*-2a_na_{n+1}^*)g_{n,TM}^m g_{n+1,TM}^{m*} \\
& +(b_n+b_{n+1}^*-2b_nb_{n+1}^*)g_{n,TE}^m g_{n+1,TE}^{m*}] \\
& +m\frac{2n+1}{n^2(n+1)^2}\frac{(n+|m|)!}{(n-|m|)!} \\
& \text{Re}[i(2a_nb_n^*-a_n-b_n^*)g_{n,TM}^m g_{n,TE}^{m*})]\}
\end{aligned}
\tag{7}
$$

2.2 Axisymmetric beams of the first kind.

In Eqs.3-6, BSCs appear in terms of the form $g_n^p g_m^{p+1*}$. Therefore, Eq.1 implies:

$$C_{pr,x} = C_{pr,y} = 0 \tag{8}$$

meaning that there is no transverse forces as we might have expected for a particle located on the axis of an axisymmeteric beam. We therefore just have to deal with Eq.7 for longitudinal forces. From Eq.1, we only have to retain $m = \pm 1$. Then, Eq.7 reduces to:

$$
\begin{aligned}
C_{pr,z} \quad = \quad & \frac{\lambda^2}{\pi}\sum_{n=1}^{\infty}\{\frac{n(n+2)}{n+1} \\
& \text{Re}[(a_n+a_{n+1}^*-2a_na_{n+1}^*)(g_{n,TM}^1 g_{n+1,TM}^{1*}+g_{n,TM}^{-1}g_{n+1,TM}^{-1*}) \\
& +(b_n+b_{n+1}^*-2b_nb_{n+1}^*)(g_{n,TE}^1 g_{n+1,TE}^{1*}+g_{n,TE}^{-1}g_{n+1,TE}^{-1*})] \\
& +\frac{2n+1}{n(n+1)}\text{Re}[i(2a_nb_n^*-a_n-b_n^*)(g_{n,TM}^1 g_{n,TE}^{1*}-g_{n,TM}^{-1}g_{n,TE}^{-1*})]\}
\end{aligned}
\tag{9}
$$

Next, we use again Eq.1 to express the result in terms of uni-index BSCs g_n, and notice that $\varepsilon^2 = 1$, leading to:

$$C_{pr,z} = \frac{\lambda^2(1+KK^*)}{4\pi}\sum_{n=1}^{\infty}\{\frac{n(n+2)}{n+1}\,\mathrm{Re}[g_n g^*_{n+1}(a_n + a^*_{n+1} - 2a_n a^*_{n+1} + b_n + b^*_{n+1} - 2b_n b^*_{n+1})] + \frac{2n+1}{n(n+1)}\varepsilon\,\mathrm{Re}[g_n g^*_n(2a_n b^*_n - a_n - b^*_n)]\} \quad (10)$$

2.3 Rayleigh limit of GLMT.

Eq.10 involves an infinite number of partial waves. However, for Rayleigh particles, we only have to retain the $(n = 1)$-partial wave term in the summation, that is to say the terms which involve only the Mie coefficient a_1, e.g. [13], [21]. This is due to the expressions of the Mie coefficients in terms of the size parameter α and the necessity, due to the assumption that we are dealing with Rayleigh particles, to retain only the terms of lowest-order powers of α. Indeed, we have [28], pp. 143-144:

$$a_1 = \frac{2i}{3}\frac{n_p^2 - 1}{n_p^2 + 2}\alpha^3 + O(i\alpha^5) + \frac{4}{9}(\frac{n_p^2 - 1}{n_p^2 + 2})^2\alpha^6 \quad (11)$$

$$b_1 = O(i\alpha^5) \quad (12)$$

in which n_p denotes the refractive index of the particle (here taken to be real) with respect to the surrounding medium, and α is the size parameter equal to $\pi d/\lambda$. The other Mie coefficients a_n and b_n $(n > 1)$ involve still higher powers of α. Real parts of a_1 are then proportional to α^6 and imaginary parts are proportional to α^3 while higher powers are discarded. Therefore, we only retain:

$$\mathrm{Im}(a_1) = \frac{2}{3}\frac{{n_p}^2 - 1}{{n_p}^2 + 2}\alpha^3 \quad (13)$$

$$\mathrm{Re}(a_1) = \frac{4}{9}(\frac{{n_p}^2 - 1}{{n_p}^2 + 2})^2\alpha^6 \quad (14)$$

Then, we still have $C_{pr,x} = C_{pr,y} = 0$, see Eq.8, while Eq.10 reduces to:

$$C_{pr,z} = \frac{3\lambda^2}{8\pi}(1+KK^*)\,\mathrm{Re}[a_1 g_1(g^*_2 - \varepsilon g^*_1)] \quad (15)$$

We then use:

$$\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Im}(z_1)(z_2) \tag{16}$$

and Eqs.13-14 to separate the α^6- and α^3-terms according to:

$$C^{\mathrm{Re}}_{pr,z} = \frac{\lambda^2}{6\pi}(1+KK^*)(\frac{m^2-1}{m^2+2})^2\alpha^6\,\mathrm{Re}\,g_1(g_2^* - \varepsilon g_1^*) \tag{17}$$

$$C^{\mathrm{Im}}_{pr,z} = \frac{-\lambda^2}{4\pi}(1+KK^*)\frac{m^2-1}{m^2+2}\alpha^3\,\mathrm{Im}\,g_1(g_2^* - \varepsilon g_1^*) \tag{18}$$

Eqs.17-18 express the longitudinal optical forces (cross-sections) exerted by (non dark) axisymmetric beams of the first kind on Rayleigh particles (while the trivial Eq.8 corresponds to transverse forces). It is seen that longitudinal forces are the summation of two kinds of forces, namely forces proportional to α^6 (traditionally associated with scattering forces in the dipole theory of forces, a statement which will have to be refined later) and forces proportional to α^3 (associated with gradient forces in the dipole theory of forces, as we shall confirm later). The interpretation of these forces requires to examine the property of the Poynting vector in the case of Rayleigh particles illuminated by axisymmetric beams of the first kind.

3 Poynting vector.

3.1 General expressions for an arbitrary location in space.

General expressions for the Poynting vector for arbitrary location in space are available from [25]. For the transverse components, we have (still using the normalization condition $E_0 H_0^*/2$) in terms of spherical coordinates r, θ, φ:

$$\begin{aligned} S_x &= \mathrm{Re}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\sum_{p=1}^{\infty}\sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \\ &\quad [\frac{k\sin\varphi}{r}(\psi_n^{''}+\psi_n)A_{nmpq} + \frac{ik\cos\theta\cos\varphi}{r}(\psi_n^{''}+\psi_n)B_{nmpq} + \frac{i\sin\theta\cos\varphi}{r^2}C_{nmpq}] \end{aligned} \tag{19}$$

$$S_y = \text{Re}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\sum_{p=1}^{\infty}\sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \tag{20}$$
$$[-\frac{k\cos\varphi}{r}(\psi_n^{''}+\psi_n)A_{nmpq} + \frac{ik\cos\theta\sin\varphi}{r}(\psi_n^{''}+\psi_n)B_{nmpq} + \frac{i\sin\theta\sin\varphi}{r^2}C_{nmpq}]$$

in which c_n^{pw} (with "pw" standing for "plane wave") are plane wave coefficients which naturally and conveniently have been introduced in the Bromwich version of the Lorenz-Mie theory [29], reading as:

$$c_n^{pw} = \frac{1}{ik}(-i)^n \frac{2n+1}{n(n+1)} \tag{21}$$

and:

$$A_{nmpq} = [\alpha_{nmpq}\psi_p^{'}\tau_p^{|q|} + q\beta_{nmpq}\psi_p\pi_p^{|q|}]P_n^{|m|} \tag{22}$$

$$B_{nmpq} = [-q\alpha_{nmpq}\psi_p^{'}\pi_p^{|q|} - \beta_{nmpq}\psi_p\tau_p^{|q|}]P_n^{|m|} \tag{23}$$

$$\begin{aligned} C_{nmpq} = \; & \alpha_{nmpq}(m\psi_n^{'}\psi_p^{'}\pi_n^{|m|}\tau_p^{|q|} + q\psi_n\psi_p\tau_n^{|m|}\pi_p^{|q|}) \\ & +\beta_{nmpq}(mq\psi_n^{'}\psi_p\pi_n^{|m|}\pi_p^{|q|} - \psi_n\psi_p^{'}\tau_n^{|m|}\tau_p^{|q|}) \end{aligned} \tag{24}$$

in which :

$$\alpha_{nmpq} = g_{p,TM}^{q*}g_{n,TE}^{m} - g_{n,TM}^{m}g_{p,TE}^{q*} \tag{25}$$

$$\beta_{nmpq} = g_{n,TM}^{m}g_{p,TM}^{q*} + g_{p,TE}^{q*}g_{n,TE}^{m} \tag{26}$$

Furthermore, ψ_n denotes Riccati-Bessel functions with the argument kr omitted for convenience, a prime denotes a derivative of a function with respect to the argument (and a double prime a double derivative), and π_n^m, τ_n^m, with argument $\cos\theta$ omitted as well for convenience, are generalized Legendre functions defined according to:

$$\pi_n^m(\cos\theta) = \frac{P_n^m(\cos\theta)}{\sin\theta} \tag{27}$$

$$\tau_n^m(\cos\theta) = \frac{dP_n^m(\cos\theta)}{d\theta} \tag{28}$$

in which P_n^m are associated Legendre functions here defined according to Hobson's convention [30]:

$$P_n^m(\cos\theta) = (-1)^m(\sin\theta)^m \frac{d^m P_n(\cos\theta)}{(d\cos\theta)^m} \tag{29}$$

in which $P_n(\cos\theta)$ are the Legendre polynomials.

For the longitudinal component, we shall start from [25]:

$$\begin{aligned} S_z &= \frac{-1}{r^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\sum_{p=1}^{\infty}\sum_{q=-p}^{+p} i c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \\ &\quad (\sin\theta S_{np}^{mq} + \cos\theta C_{np}^{mq}) \end{aligned} \tag{30}$$

in which:

$$\begin{aligned} S_{np}^{mq} &= kr[-g_{n,TM}^m g_{p,TM}^{q*}\psi_p(\psi_n + \psi_n^{''})P_n^{|m|}\tau_p^{|q|} \\ &\quad +g_{n,TE}^m g_{p,TE}^{q*}\psi_n(\psi_p + \psi_p^{''})P_p^{|q|}\tau_n^{|m|} \\ &\quad +q g_{n,TM}^m g_{p,TE}^{q*}\psi_p^{'}(\psi_n + \psi_n^{''})P_n^{|m|}\pi_p^{|q|} \\ &\quad +m g_{n,TM}^m g_{p,TE}^{q*}\psi_n^{'}(\psi_p + \psi_p^{''})P_p^{|q|}\pi_n^{|m|}] \end{aligned} \tag{31}$$

$$\begin{aligned} C_{np}^{mq} &= -g_{n,TM}^m g_{p,TM}^{q*}\psi_p\psi_n^{'}(\tau_n^{|m|}\tau_p^{|q|} + mq\pi_n^{|m|}\pi_p^{|q|}) \\ &\quad +g_{n,TM}^m g_{p,TE}^{q*}\psi_n^{'}\psi_p^{'}(m\pi_n^{|m|}\tau_p^{|q|} + q\pi_p^{|q|}\tau_n^{|m|}) \\ &\quad -g_{n,TE}^m g_{p,TM}^{q*}\psi_p\psi_n(m\pi_n^{|m|}\tau_p^{|q|} + q\pi_p^{|q|}\tau_n^{|m|}) \\ &\quad +g_{n,TE}^m g_{p,TE}^{q*}\psi_n\psi_p^{'}(mq\pi_n^{|m|}\pi_p^{|q|} + \tau_n^{|m|}\tau_p^{|q|}) \end{aligned} \tag{32}$$

3.2 Axisymmetric beams of the first kind for an arbitrary location in space.

For axisymmetridc beams of the first kind, Eq.1 shows that we have only to retain BSCs with $m = \pm 1$. Therefore, the strings of subscripts $nmpq$ in Eqs.22-26 reduce to $n1p1$, $n1p-1$, $n-1p1$ and $n-1p-1$. Using Eqs.25-26 and expressing the bi-index BSCs in terms of uni-index BSCs using Eq.1, we find that the non-zero terms involved in these equations are those with the strings $n1p1$ and $n-1p-1$. These terms are found to read as:

$$\alpha_{n1p1} = i\varepsilon\beta_{n1p1} = \frac{i\varepsilon}{2} g_n g_p^* \tag{33}$$

$$\alpha_{n-1p-1} = -i\varepsilon\beta_{n-1p-1} = \frac{-i\varepsilon KK^*}{2} g_n g_p^* \tag{34}$$

The non-zero coefficients A_{nmpq}, B_{nmpq} and C_{nmpq} of Eqs.22-24 are then found to be:

$$A_{n1p1} = -A_{n-1p-1}/KK^* = \frac{g_n g_p^*}{2}(i\varepsilon\psi'_p\tau_p^1 + \psi_p\pi_p^1)P_n^1 \tag{35}$$

$$B_{n1p1} = B_{n-1p-1}/KK^* = \frac{-g_n g_p^*}{2}(i\varepsilon\psi'_p\pi_p^1 + \psi_p\tau_p^1)P_n^1 \tag{36}$$

$$\begin{aligned} C_{n1p1} &= C_{n-1p-1}/KK^* = \frac{g_n g_p^*}{2}[i\varepsilon(\psi'_n\psi'_p\pi_n^1\tau_p^1 + \psi_n\psi_p\tau_n^1\pi_p^1) \\ &\quad +\psi'_n\psi_p\pi_n^1\pi_p^1 - \psi_n\psi'_p\tau_n^1\tau_p^1] \end{aligned} \tag{37}$$

in which arguments are omitted for convenience. Inserting Eqs.35-37 into Eq.19, when the only allowed strings of subscripts are $n1p1$ and $n-1p-1$, we obtain:

$$\begin{aligned} S_x &= \frac{1}{2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} g_n g_p^* \\ &\quad \{\frac{k\sin\varphi}{r}(\psi''_n + \psi_n)(1-KK^*)(i\varepsilon\psi'_p\tau_p^1 + \psi_p\pi_p^1)P_n^1 \\ &\quad -\frac{ik\cos\theta\cos\varphi}{r}(\psi''_n + \psi_n)(1+KK^*)(i\varepsilon\psi'_p\pi_p^1 + \psi_p\tau_p^1)P_n^1 \\ &\quad +\frac{i\sin\theta\cos\varphi}{r^2}(1+KK^*)[i\varepsilon(\psi'_n\psi'_p\pi_n^1\tau_p^1 + \psi_n\psi_p\tau_n^1\pi_p^1) \\ &\quad +\psi'_n\psi_p\pi_n^1\pi_p^1 - \psi_n\psi'_p\tau_n^1\tau_p^1]\} \end{aligned} \tag{38}$$

Eqs.19-20 show that we pass from S_x to S_y by changing $\sin\varphi$ to $(-\cos\varphi)$ and $\cos\varphi$ to $\sin\varphi$. Therefore, we immediately have:

$$\begin{aligned} S_y &= \frac{1}{2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} g_n g_p^* \\ &\{\frac{-k\cos\varphi}{r}(\psi_n'' + \psi_n)(1 - KK^*)(i\varepsilon\psi_p'\tau_p^1 + \psi_p\pi_p^1)P_n^1 \\ &-\frac{ik\cos\theta\sin\varphi}{r}(\psi_n'' + \psi_n)(1 + KK^*)(i\varepsilon\psi_p'\pi_p^1 + \psi_p\tau_p^1)P_n^1 \\ &+\frac{i\sin\theta\sin\varphi}{r^2}(1 + KK^*)[i\varepsilon(\psi_n'\psi_p'\pi_n^1\tau_p^1 + \psi_n\psi_p\tau_n^1\pi_p^1) \\ &+\psi_n'\psi_p\pi_n^1\pi_p^1 - \psi_n\psi_p'\tau_n^1\tau_p^1]\} \end{aligned} \tag{39}$$

Similarly, recalling that only superscripts $m = \pm 1$ are to be retained, Eq.30 becomes:

$$\begin{aligned} S_z &= \frac{-1}{r^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*}\{e^{-2i\varphi}(\sin\theta S_{np}^{-11} + \cos\theta C_{np}^{-11}) \\ &+\sin\theta(S_{np}^{-1-1} + S_{np}^{11}) + \cos\theta(C_{np}^{-1-1} + C_{np}^{11}) \\ &+e^{2i\varphi}(\sin\theta S_{np}^{1-1} + \cos\theta C_{np}^{1-1})\} \end{aligned} \tag{40}$$

that we do not need to work out extensively.

3.3 Axisymmetric beams of the first kind at the particle location.

Next, the Poynting vector needs to be evaluated at the particle location. This particle location is designated by the subscript P. Formally, this particle location can be reached by first taking an axis location ($\theta = 0$) followed by $r = 0$. Therefore, "P" below is equivalent to "$\theta = 0, r = 0$". From Eqs.38-39, we then obtain:

$$\begin{aligned} (S_x)_P &= \frac{1}{2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} g_n g_p^* \\ &[\frac{k\sin\varphi}{r}(\psi_n'' + \psi_n)(1 - KK^*)(i\varepsilon\psi_p'\tau_p^1 + \psi_p\pi_p^1)P_n^1 \\ &-\frac{ik\cos\varphi}{r}(\psi_n'' + \psi_n)(1 + KK^*)(i\varepsilon\psi_p'\pi_p^1 + \psi_p\tau_p^1)P_n^1]_P \end{aligned} \tag{41}$$

$$(S_y)_P = \frac{1}{2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} g_n g_p^* \tag{42}$$

$$[\frac{-k\cos\varphi}{r}(\psi_n'' + \psi_n)(1 - KK^*)(i\varepsilon\psi_p'\tau_p^1 + \psi_p\pi_p^1)P_n^1$$

$$-\frac{ik\sin\varphi}{r}(\psi_n'' + \psi_n)(1 + KK^*)(i\varepsilon\psi_p'\pi_p^1 + \psi_p\tau_p^1)P_n^1]_P$$

Both Eqs.41-42 contain the term $(P_n^1)_P$. We however may establish that, e.g. Eq.(40) in [25]:

$$P_n^{|m|}(\theta = 0) = \delta_{|m|0} \tag{43}$$

Therefore, $(P_n^1)_P = 0$ and we obtain:

$$(S_x)_P = (S_y)_P = 0 \tag{44}$$

which is once again what we should have expected. The same result is obtained from Eqs.(41)-(46) from [25] because, due to Eq.1, the g_n^0's are zero so that μ_{np}, ν_{np}, η_{np} and λ_{np} of Eqs.(42), (43), (45) and (46) of [25] respectively are zero.

For the longitudinal component, we start from Eq.40 and, taking the value on the axis for $\theta = 0$, we obtain:

$$(S_z)_P = \frac{-1}{r^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*}[e^{-2i\varphi}C_{np}^{-11} + (C_{np}^{-1-1} + C_{np}^{11}) + e^{2i\varphi}C_{np}^{1-1}]_P \tag{45}$$

We expect that the φ-dependent term of Eq.45 should be zero, according to the definition of an axisymmetric beam. To check this, we isolate this term to work it out independently, denoting it as $(S_z)_{P\varphi}$. We readily have:

$$(S_z)_{P\varphi} = \frac{-1}{r^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*}[\cos(2\varphi)(C_{np}^{-11} + C_{np}^{1-1}) + i\sin(2\varphi)(C_{np}^{1-1} - C_{np}^{-11})]_P \tag{46}$$

We now recall that, see Eq.(39) in [25]:

$$\pi_n^1(\theta=0) = \tau_n^1(\theta=0) = \Omega_n = \frac{-n(n+1)}{2} \tag{47}$$

Using Eq.32, we immediately obtain:

$$C_{np}^{-11} = C_{np}^{1-1} = 0 \tag{48}$$

so that we indeed have $(S_z)_{P\varphi} = 0$ as expected, while $(S_z)_P$ reduces to:

$$(S_z)_P = \frac{-1}{r^2} \,\mathrm{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*} (C_{np}^{-1-1} + C_{np}^{11})_P \tag{49}$$

To evaluate $(C_{np}^{-1-1})_P$ and $(C_{np}^{11})_P$, we (i) use Eq.32 to express them in terms of bi-index BSCs, then (ii) use Eq.1 to express them in terms of uni-index BSCs, then (iii) express the involved generalized Legendre functions for $\theta = 0$ using Eq.47. This leads to:

$$C_{np}^{11} = C_{np}^{-1-1}/KK^* = \frac{np(n+1)(p+1)}{8} g_n g_p^* [\psi_n \psi_p^{'} - \psi_p \psi_n^{'} - i\varepsilon(\psi_n' \psi_p' + \psi_p \psi_n)]_P \tag{50}$$

which, once inserted into Eq.49, leads to:

$$\begin{aligned} (S_z)_P \quad &= \quad \frac{-1}{8}(1+KK^*) \,\mathrm{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} i k^2 c_n^{pw} c_p^{pw*} np(n+1)(p+1) g_n g_p^* \\ &\times [\frac{\psi_n \psi_p^{'} - \psi_p \psi_n^{'} - i\varepsilon(\psi_n' \psi_p' + \psi_p \psi_n)}{k^2 r^2}]_P \end{aligned} \tag{51}$$

Now, we use Eq.21, to obtain:

$$c_1^{pw} c_1^{pw*} = \frac{9}{4k^2} \tag{52}$$

Furthermore, we have :

$$\psi_n(x) = x j_n(x) \tag{53}$$

$$\psi_n'(x) = (n+1)j_n(x) - xj_{n+1}(x) \tag{54}$$

and, e.g. Eq.(11.144) in [31]:

$$j_n(x) = 2^n x^n \sum_{s=0}^{\infty} \frac{(-1)^s (s+n)!}{s!(2s+2n+1)!} x^{2s} \tag{55}$$

Then, we use Eq.53 to establish:

$$[\frac{\psi_n(kr)}{kr}]_{r=0} = [j_n(kr)]_{r=0} = \delta_{n0} \tag{56}$$

and we use Eqs.53, 54, and 56 to establish:

$$[\frac{\psi_n'(kr)}{kr}]_{r=0} = \frac{2}{3}\delta_{n1} \tag{57}$$

Inserting Eqs.52, 56, 57 into Eq.51 finally leads to the following simple expression:

$$(S_z)_P = \frac{-1}{2}\varepsilon(1+KK^*)\left|g_1\right|^2 \tag{58}$$

It is interesting to remark that this result is valid in the case of Rayleigh particles, although we did not need to introduce any assumption on the size of the particle as we needed to work out the expressions of the optical forces. This is because the size of the particle does not intervene in the concept of Poynting vector. In other words, evaluating S_z at P is directly a "point-like" evaluation in the same way that a Rayleigh particle is a "point-like" particle.

3.4 Restricted Poynting vector at arbitrary location.

It is interesting to remark that Eq.58 may be reached following another path, using a restricted Poynting vector in which only $(n=1)$ partial waves are retained, this restriction being inspired by the fact that Rayleigh particles are not sensitive to higher-order partial waves. This alternative way of deriving Eq.58 will allow one to emphasize the particular role played by the lowest-order partial waves. Let us then consider Eq.38, retain only the $(n=p=1)$-terms in the summation, use Eq.52, and evaluate $P_1^1 = -\sin\theta$, $\pi_1^1 = -1$ and $\tau_1^1 = -\cos\theta$. We then obtain:

$$\begin{aligned} S_x &= \frac{9}{8k^2}\left|g_1\right|^2 \sin\theta\{\frac{k}{r}(\psi_1'' + \psi_1)[(1 - KK^*)\sin\varphi\psi_1 \\ &+\varepsilon(1 + KK^*)\cos\theta\cos\varphi\psi_1'] \\ &-\frac{\varepsilon\cos\theta\cos\varphi}{r^2}(1 + KK^*)(\psi_1'\psi_1' + \psi_1\psi_1)\} \end{aligned} \tag{59}$$

We then remember that we pass from S_x to S_y by changing $\sin\varphi$ to $(-\cos\varphi)$ and $\cos\varphi$ to $\sin\varphi$. Therefore, from Eq.59, we immediately obtain:

$$\begin{aligned} S_y &= \frac{9}{8k^2}\left|g_1\right|^2 \sin\theta\{\frac{k}{r}(\psi_1'' + \psi_1)[(KK^* - 1)\cos\varphi\psi_1 \\ &+\varepsilon(1 + KK^*)\cos\theta\sin\varphi\psi_1'] \\ &-\frac{\varepsilon\cos\theta\sin\varphi}{r^2}(1 + KK^*)(\psi_1'\psi_1' + \psi_1\psi_1)\} \end{aligned} \tag{60}$$

For S_z, we start from Eq.40 and, again retaining only the $(n = p = 1)$-terms, we have :

$$\begin{aligned} S_z &= \frac{-1}{r^2}\,\mathrm{Re}\, ic_1^{pw}c_1^{pw*}\{e^{-2i\varphi}(\sin\theta S_{11}^{-11} + \cos\theta C_{11}^{-11}) \\ &+\sin\theta(S_{11}^{-1-1} + S_{11}^{11}) + \cos\theta(C_{11}^{-1-1} + C_{11}^{11}) \\ &+e^{2i\varphi}(\sin\theta S_{11}^{1-1} + \cos\theta C_{11}^{1-1})\} \end{aligned} \tag{61}$$

Then, we evaluate the various coefficients S_{np}^{mq} and C_{np}^{mq} involved in Eq.61, using Eqs.31-32, evaluating the involved generalized Legendre polynomials as previously, and expressing the bi-index BSCs in terms of uni-index BSCs using Eq.1. Once these coefficients are inserted into Eq.61, we obtain:

$$S_{11}^{-11} = \frac{-K}{2}kr\sin\theta\cos\theta(\psi_1'' + \psi_1)\psi_1\left|g_1\right|^2 \tag{62}$$

$$S_{11}^{-1-1} = \frac{-i\varepsilon KK^*}{2}kr\sin\theta(\psi_1'' + \psi_1)\psi_1'\left|g_1\right|^2 \tag{63}$$

$$S_{11}^{11} = \frac{-i\varepsilon}{2} kr \sin\theta (\psi_1'' + \psi_1)\psi_1' \left|g_1\right|^2 \tag{64}$$

$$S_{11}^{1-1} = \frac{-K^*}{2} kr \sin\theta \cos\theta (\psi_1'' + \psi_1)\psi_1 \left|g_1\right|^2 \tag{65}$$

$$C_{11}^{-11} = \frac{K}{2} \sin^2\theta \psi_1' \psi_1 \left|g_1\right|^2 \tag{66}$$

$$C_{11}^{-1-1} = \frac{-i\varepsilon K K^*}{2} \cos\theta (\psi_1'\psi_1' + \psi_1\psi_1) \left|g_1\right|^2 \tag{67}$$

$$C_{11}^{11} = \frac{-i\varepsilon}{2} \cos\theta (\psi_1'\psi_1' + \psi_1\psi_1) \left|g_1\right|^2 \tag{68}$$

$$C_{11}^{1-1} = \frac{K^*}{2} \sin^2\theta \psi_1' \psi_1 \left|g_1\right|^2 \tag{69}$$

Eqs.62-69 are then inserted in Eq.61 which is then found to reduce to:

$$S_z = \frac{-9\varepsilon}{8k^2r^2}(KK^* + 1)[kr \sin^2\theta (\psi_1'' + \psi_1)\psi_1' + \cos^2\theta (\psi_1'\psi_1' + \psi_1\psi_1)] \left|g_1\right|^2 \tag{70}$$

in which we have used Eq.52. This result does not depend any more on φ as it should.

3.5 Restricted Poynting vector at the particle location.

From Eqs.59-60, we observe that S_x and S_y are proportional to $\sin\theta$. Therefore, at the particle location (more generally on the z-axis), for which $\theta = 0$, we have:

$$(S_x)_P = (S_y)_P = 0 \tag{71}$$

in which we recover Eq.44 as it should. For S_z, Eq.70 similarly reduces to:

$$(S_z)_{\theta=0} = \frac{-9\varepsilon}{8k^2r^2}(KK^* + 1)(\psi_1'\psi_1' + \psi_1\psi_1)\,|g_1|^2 \tag{72}$$

and, finally, using Eqs.56-57, we obtain:

$$(S_z)_P = \frac{-\varepsilon}{2}(KK^* + 1)\,|g_1|^2 \tag{73}$$

which is identical to Eq.58.

4 Optical forces for on-axis Gaussian beams.

4.1 Generalities.

Among the infinite set of on-axis circularly symmetric Bessel functions, the only case corresponding to a (non dark) axisymmetric beam of the first kind is the zeroth-order Bessel beam (which is a non vortex beam). This case has been discussed in [23] and is not considered any more in the present paper. We then rather study the case of Gaussian beams which has been discussed by Lock [21] in the limit of weak beam confinement. As we shall see, such a limit actually washes out the occurrence of what we call non-standard forces, for the time being.

One of the difficulties with Gaussian beams is to possess a beam description which perfectly satisfies Maxwell's equations (that we call a Maxwellian description). Indeed, the most used general description of Gaussian beams may be the one introduced by Davis [32] in 1979. In this formulation [33], and [3], pp.97-106, we use a linearly vector potential $\mathbf{A} = (A_x, 0, 0)$ whose non zero component reads as:

$$A_x = \psi(x, y, z)\exp(-ikz) \tag{74}$$

Next, we introduce the beam confinement parameter $s = 1/(kw_0)$ in which w_0 is the beam waist radius, and we expand the function ψ in powers of s^2 according to:

$$\psi = \psi_0 + s^2\psi_2 + s^4\psi_4 + ... \tag{75}$$

The lowest-order term ψ_0, which is sufficient to afterward recursively determine the higher-order modes, represents the fundamental mode of the

Gaussian beam. For some reason fairly subtle to explain (but see [33]), the fields restricted to ψ_0 define the first-order Davis beam, while the higher-order modes are named third-order, fifth-order Davis beams and so on. Explicit expressions for the kth-order Davis beams are known only up to $k = 5$ [34]. None of these beams is an exact solution of Maxwell's equations. Maxwell's equations are satisfied only in the limit $k \to \infty$. We must also have in mind that s is a small parameter. For instance, a typical figure is $s \simeq 10^{-3}$ for $\lambda = 0.5$ μm and $w_0 = 50$ μm. It is 0 for a plane wave while its maximal value is $s \simeq 1/(2\pi) \simeq 0.16$ for a tightly focused beam with $w_0 \simeq \lambda$.

Nevertheless, Davis formulation has been the basis to the development of Maxwellian descriptions of Gaussian beams, named (i) localized approximation (LA), (ii) modified localized approximation (MLA), and (iii) improved standard beam description (ISBD). These Maxwellian descriptions are now going to be studied systematically from the point of view of optical forces in the framework of the Rayleigh limit of GLMT. In all these beams, the parameters K and ε introduced in Eq.1 receive the values $K = 1$ and $\varepsilon = -1$ [24], [25]. Eqs.17, 18, and 58, 73 then simplifies to:

$$C^{\mathrm{Re}}_{pr,z} = \frac{\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6 \,\mathrm{Re}\, g_1(g_1^* + g_2^*) \tag{76}$$

$$C^{\mathrm{Im}}_{pr,z} = \frac{-\lambda^2}{2\pi}\frac{m^2-1}{m^2+2}\alpha^3 \,\mathrm{Im}\, g_1(g_1^* + g_2^*) \tag{77}$$

$$(S_z)_P = |g_1|^2 \tag{78}$$

Insofar as the transverse expressions are trivial, i.e. $C_{pr,x} = C_{pr,y} = 0$ and $S_x = S_y = 0$, we shall focus ourselves on the longitudinal quantities expressed by Eqs.76-78.

4.2 Localized approximation.

The localized approximation to the evaluation of BSCs has been an invaluable tool to carry out GLMT computations. It has been developed in [35], [36], [37], [38] for Gaussian beams and in [39] for laser sheets. It received a rigorous justification for Gaussian beams in [40], see [41] for a review. In the on-axis localized approximation framework, the uni-index BSCs read as:

$$g_n = \frac{e^{ikz_0}}{1+iA}\exp[\frac{-(n+1/2)^2 s^2}{1+iA}] \tag{79}$$

in which:

$$A = 2s\frac{z_0}{w_0} = 2kz_0 s^2 \tag{80}$$

in which we have used the definition of the beam confinement factor, and in which z_0 defines the distance of the beam waist center with respect to the particle and is then 0 for a beam waist location (in which case A is zero as well).

From Eqs.76-78, we need:

$$g_1 = \frac{e^{ikz_0}}{1+iA}\exp(\frac{-9s^2/4}{1+iA}) \tag{81}$$

$$g_2 = \frac{e^{ikz_0}}{1+iA}\exp[\frac{-25s^2/4}{1+iA}] \tag{82}$$

We then evaluate:

$$g_1(g_1^* + g_2^*) = \frac{1}{1+A^2}[\mathcal{M} + \mathcal{N}(\cos\Theta - i\sin\Theta)] \tag{83}$$

in which:

$$\mathcal{M} = \exp(\frac{-9s^2/2}{1+4s^2\frac{z_0^2}{w_0^2}}) \tag{84}$$

$$\mathcal{N} = \exp(\frac{-17s^2/2}{1+A^2}) \tag{85}$$

$$\Theta = \frac{8ks^4 z_0}{1+A^2} \tag{86}$$

Let us first explore the **weak confinement limit** when $s \to 0$. Then we have $\mathcal{M} \approx 1$, $\mathcal{N} \approx 1$, $\cos\Theta \approx 1$ and $\sin\Theta \approx \Theta$, leading to:

$$g_1(g_1^* + g_2^*) = \frac{1}{1+A^2}[2 - i\Theta)] \tag{87}$$

Inserting Eq.87 into Eqs.76-77, we obtain:

$$C^{\mathrm{Re}}_{pr,z} = \frac{2\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6\frac{1}{1+A^2} \tag{88}$$

$$C^{\mathrm{Im}}_{pr,z} = \frac{4k\lambda^2}{\pi}\frac{m^2-1}{m^2+2}\alpha^3\frac{s^4 z_0}{(1+A^2)^2} \tag{89}$$

Relaxing the weak confinement assumption, we may rewrite $\mathrm{Re}\, g_1(g_1^* + g_2^*)$ as a sum of two terms according to:

$$\mathrm{Re}\, g_1(g_1^* + g_2^*) = \mathrm{Re}\, |g_1|^2 + \mathrm{Re}\, g_1 g_2^* \tag{90}$$

which induces the splitting of $C^{\mathrm{Re}}_{pr,z}$ into two terms, according to:

$$C^{\mathrm{Re}}_{pr,z} = C^{\mathrm{Re},S}_{pr,z} + C^{\mathrm{Re},NS}_{pr,z} \tag{91}$$

in which:

$$C^{\mathrm{Re},S}_{pr,z} = \frac{\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6\frac{1}{1+A^2}\exp[\frac{-9s^2}{2(1+A^2)}] \tag{92}$$

which is half $C^{\mathrm{Re}}_{pr,z}$ of Eq.88 in the beam confinement limit, and:

$$C^{\mathrm{Re},NS}_{pr,z} = \frac{\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6\frac{1}{1+A^2}\exp[\frac{-17s^2}{2(1+A^2)}]\cos(\frac{4s^2}{1+A^2}) \tag{93}$$

It is interesting to remark that, in the weak confinement limit, Eq.93 reduces as well to half $C^{\mathrm{Re}}_{pr,z}$ of Eq.88. Therefore, in this limit, the summation of the S-term of Eq.92 and of the NS-term of Eq.93 is equal to the total $C^{\mathrm{Re}}_{pr,z}$ of Eq.88. This fact is related to the other fact that $C^{\mathrm{Re},S}_{pr,z}$ and $C^{\mathrm{Re},NS}_{pr,z}$ have the same order of magnitude in the present LA-case, in contrast with we shall observe for MLA and ISBD, e.g. Eqs.113 and 123.

Next, we evaluate, neglecting higher-order terms:

$$\mathcal{M} = \exp(\frac{-9s^2/2}{1+A^2}) \approx 1 - \frac{9s^2/2}{1+A^2} + O(s^4) \tag{94}$$

$$\mathcal{N} = \exp(\frac{-17s^2/2}{1+A^2}) \approx 1 - \frac{17s^2/2}{1+A^2} + O(s^4) \tag{95}$$

$$\cos\Theta \approx 1 + O(s^6) \tag{96}$$

$$\sin\Theta \approx \frac{8ks^4 z_0}{1+A^2} + O(s^{12}) \tag{97}$$

Inserting these results in Eq.83, we obtain, neglecting higher-order terms:

$$g_1(g_1^* + g_2^*) \simeq \frac{1}{1+A^2}[2 - \frac{13s^2}{1+A^2} - i\frac{8ks^4 z_0}{1+A^2}] \tag{98}$$

Therefore, for $C_{pr,z}^{\mathrm{Im}}$ of Eq.77, we obtain, using Eq.98, and again neglecting higher-order terms:

$$C_{pr,z}^{\mathrm{Im}} = \frac{4k\lambda^2}{\pi}\frac{m^2-1}{m^2+2}\alpha^3\frac{s^4 z_0}{(1+A^2)^2} \tag{99}$$

This result introduces an $O(s^4)$ ratio for $C_{pr,z}^{\mathrm{Im}}/C_{pr,z}^{\mathrm{Re},S}$ or $C_{pr,z}^{\mathrm{Im}}/C_{pr,z}^{\mathrm{Re},NS}$.

4.3 Modified localized approximation.

The modified localized approximation has been introduced in [40], [42]. It introduces a slight modification of the localized approximation. Further discussions of LA and MLA are available from [43] and [44]. In this framework, the uni-index BSCs read as:

$$g_n = \frac{e^{ikz_0}}{1+iA}\exp[\frac{-(n-1)(n+2)s^2}{1+iA}] \tag{100}$$

Then, processing similarly as for the LA-case, we have:

$$g_1 = \frac{e^{ikz_0}}{1+iA} \tag{101}$$

$$g_2 = \frac{e^{ikz_0}}{1+iA} \exp[\frac{-4s^2}{1+iA}] \tag{102}$$

$$g_1(g_1^* + g_2^*) = \frac{1}{1+A^2}[1 + \mathcal{K}(\cos\Theta - i\sin\Theta)] \tag{103}$$

in which Θ is defined by Eq.86 and:

$$\mathcal{K} = \exp(\frac{-4s^2}{1+A^2}) \tag{104}$$

In the **weak confinement limit** when $s \to 0$, we have $\mathcal{K} \approx 1$, $\cos\Theta \approx 1$ and $\sin\Theta \approx \Theta$, leading to:

$$g_1(g_1^* + g_2^*) = \frac{1}{1 + 4s^2\frac{z_0^2}{w_0^2}}[2 - i\Theta)] \tag{105}$$

and to:

$$C_{pr,z}^{\mathrm{Re},S} = \frac{2\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6\frac{1}{1+A^2} \tag{106}$$

$$C_{pr,z}^{\mathrm{Im}} = \frac{4k\lambda^2}{\pi}\frac{m^2-1}{m^2+2}\alpha^3\frac{s^4 z_0}{(1+A^2)^2} \tag{107}$$

which are identical to Eqs.87-89 respectively.

If we relax the weak confinement limit assumption, we have:

$$\mathcal{K} = \exp(\frac{-4s^2}{1+A^2}) \approx 1 - \frac{4s^2}{1+A^2} + O(s^4) \tag{108}$$

while Eqs.96-97 are still valid. Neglecting higher-order terms, we then obtain:

$$g_1(g_1^* + g_2^*) \simeq \frac{1}{1+A^2}[2 - \frac{4s^2}{1+A^2} \\ -i(1 - \frac{4s^2}{1+A^2})\frac{8ks^4 z_0}{1+A^2}] \tag{109}$$

It is then found that:

$$C_{pr,z}^{\mathrm{Re}} = C_{pr,z}^{\mathrm{Re},S} + C_{pr,z}^{\mathrm{Re},NS} \tag{110}$$

in which:

$$C_{pr,z}^{\mathrm{Re},S} = \frac{2\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2 \alpha^6 \frac{1}{1+A^2} \tag{111}$$

$$C_{pr,z}^{\mathrm{Re},NS} = \frac{-4\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2 \alpha^6 \frac{s^2}{(1+A^2)^2} \tag{112}$$

Therefore, $C_{pr,z}^{\mathrm{Re}}$ can again been separated into two terms. Furthermore, we find that the ratio of the NS- over the S-terms is given by:

$$\frac{C_{pr,z}^{\mathrm{Re},NS}}{C_{pr,z}^{\mathrm{Re},S}} = \frac{-2s^2}{1+4s^2\frac{z_0^2}{w_0^2}} = \frac{-2s^2}{(1+A^2)} \simeq O(s^2) \tag{113}$$

which is $O(s^2)$. For $C_{pr,z}^{\mathrm{Im}}$, we obtain, neglecting higher-order terms:

$$C_{pr,z}^{\mathrm{Im}} = \frac{4k\lambda^2}{\pi}\frac{m^2-1}{m^2+2}\alpha^3 \frac{s^4 z_0}{(1+A^2)^2} \tag{114}$$

which is identical to Eqs.89 and 99, and introduces another $O(s^2)$ ratio for $C_{pr,z}^{\mathrm{Im}}/C_{pr,z}^{\mathrm{Re},NS}$, that is to say an $O(s^4)$ ratio for $C_{pr,z}^{\mathrm{Im}}/C_{pr,z}^{\mathrm{Re},S}$.

4.4 Improved standard beams.

In [40] devoted to on-axis Gaussian beams in the Davis formulation, it was found that the uni-index BSCs g_n for the first-, third- and fifth-Davis beams could be expressed in an unified way as the sum of a first term satisfying Maxwell's equations and of a second term which was coordinate-dependent in contrast with the fact that the BSCs should not depend on any coordinate, see Eqs.(75) and (76) in [40]. These coordinate-dependent terms indicate that the $k$the-Davis beams used ($k = 1, 3, 5$) do not exactly satisfy Maxwell's equations. Removing the coordinate-dependent terms and generalizing the expression of the Maxwellian contributions allowed one to introduce what was called a S-beam (S standing for "standard"). Standard beams have been discussed and/or used as well in [8], [33], [43], [44], and have been considered as providing a "perfect" ("standard") on-axis description of Gaussian beams. Unfortunately, it has been observed [8] that the standard beam description exhibited a finite radius of convergence, therefore spoiling the possibility of evaluating radiation pressure forces, particularly reverse forces, for some interesting ranges of parameters, using GLMT. The improved standard beam description has therefore been established afterward with an infinite radius of convergence, and can then be claimed to provide an optimal Maxwellian description of Gaussian beams. In this framework, the uni-index BSCs g_n read as, e.g. Eq.(16) in [45]:

$$g_n = \frac{e^{ikz_0}}{1+iA} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{(-s^2)^p}{(1+iA)^p} \frac{(n+1+p)!}{(n-1-p)!} \frac{(n-1)!}{(n+1)!} \tag{115}$$

As explained after Eq.(2) in [45], this way to write the equation seems to make a problem when $n = p = 1$. To avoid this difficulty, we shall rewrite Eq.115 under the following form:

$$g_n = \frac{e^{ikz_0}}{1+iA} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{(-s^2)^p}{(1+iA)^p} \frac{(n-p)(n-p+1)...n(n+1)...(n+p)(n+p+1)}{n(n+1)} \tag{116}$$

Processing similarly as for LA and MLA, we then have:

$$g_1 = \frac{e^{ikz_0}}{1+iA} \tag{117}$$

$$g_2 = \frac{e^{ikz_0}}{1+iA}(1 - \frac{4s^2}{1+iA}) \tag{118}$$

leading to:

$$g_1(g_1^* + g_2^*) = \frac{1}{1+A^2}[2 - \frac{4s^2(1+iA)}{1+A^2}] \tag{119}$$

It is then once again found that $C_{pr,z}^{\mathrm{Re}}$ can be separated into two terms according to:

$$C_{pr,z}^{\mathrm{Re}} = C_{pr,z}^{\mathrm{Re},S} + C_{pr,z}^{\mathrm{Re},NS} \tag{120}$$

in which:

$$C_{pr,z}^{\mathrm{Re},S} = \frac{2\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6\frac{1}{1+A^2} \tag{121}$$

$$C_{pr,z}^{\mathrm{Re},NS} = \frac{-4\lambda^2}{3\pi}(\frac{m^2-1}{m^2+2})^2\alpha^6\frac{s^2}{(1+A^2)^2} \tag{122}$$

We then note that Eq.121 is identical to Eqs.88 and 111, and that Eq.122 is identical to with Eq.112. The ratio of NS- and S-terms is then given by:

$$\frac{C_{pr,z}^{\mathrm{Re},NS}}{C_{pr,z}^{\mathrm{Re},S}} = \frac{-2s^2}{1+4s^2\frac{z_0^2}{w_0^2}} = \frac{-2s^2}{(1+A^2)} \simeq O(s^2) \tag{123}$$

which is identical to Eq.113. As far as $C_{pr,z}^{\mathrm{Im}}$ is concerned, it is found to read as:

$$C_{pr,z}^{\mathrm{Im}} = \frac{4k\lambda^2}{\pi}\frac{m^2-1}{m^2+2}\alpha^3\frac{s^4 z_0}{(1+A^2)^2} \tag{124}$$

which is identical to Eqs.89, 94, 107, 114, and once again introduces another $O(s^2)$ ratio when compared to NS-forces and another $O(s^4)$-ratio when compared to S-forces.

5 Identification of optical forces within the framework of the dipole theory of forces.

The different kinds of forces exhibited in the previous section, and derived using the Rayleigh limit of GLMT, are now going to be interpreted through their identification with the forces of the dipole theory of forces. We shall distinguish (i) scattering forces, (ii) gradient forces, and (iii) non-standard forces.

5.1 Scattering forces.

In the framework of the dipole theory of forces, scattering forces are proportional to the sixth power of α and to the corresponding Poynting vector components. In the MLA-case, and in the weak confinement limit, it has been established by Lock [21] that the force $C_{pr,z}^{\mathrm{Re},S}$ described by Eq.106 is indeed a scattering force in the framework of the dipole theory of forces. The superscript S in $C_{pr,z}^{\mathrm{Re},S}$ may then be viewed as standing for "Scattering", or standing for "Standard", by opposition to the superscript NS in $C_{pr,z}^{\mathrm{Re},NS}$ where it stands for "Non-standard".

More generally, all forces labelled $C_{pr,z}^{\mathrm{Re},S}$ are scattering forces both proportional to the sixth power of α and to the z-component of the Poynting vector (taken at P). In the case of LA, this is obvious as exhibited by the decomposition carried out in Eq.90 when the weak confinement limit is not implemented. Furthermore, when the weak confinement limit is assumed, it happens that $C_{pr,z}^{\mathrm{Re},S}$ of Eq.92 and $C_{pr,z}^{\mathrm{Re}}$ of Eq.88 are all proportional to $1/(1+A^2)$ which is exactly the value of $(S_z)_P = |g_1|^2$ in this limit, indicating that $C_{pr,z}^{\mathrm{Re},S}$and $C_{pr,z}^{\mathrm{Re}}$ are indeed scattering forces in this limit.

For MLA and ISBD, g_1 is given by $e^{ikz_0}/(1+iA)$, see Eqs.101 and 117 respectively, so that we still have $(S_z)_P = 1/(1+A^2)$. Therefore, $C_{pr,z}^{\mathrm{Re},S}$ which are proportional to $1/(1+A^2)$, see Eqs. 106 and 121, are indeed once again scattering forces.

A complementary point of view is obtained if we return to the most general framework of the Rayleigh limit of GLMT. We then have that the forces proportional to the sixth power of α are proportional to $\mathrm{Re}(G)$, with G reading as (see Eqs.(26), (43)-(46) in [13]):

$$G = G^{11} + G^{12} + G^0 \tag{125}$$

in which we have, after implementing the conditions of Eq.1 of the present paper:

$$G^{11} = i[g_{1,TM}^{-1} g_{1,TE}^{-1*} - g_{1,TM}^{1} g_{1,TE}^{1*}] = \frac{1}{2}\,|g_1|^2 \tag{126}$$

$$G^{12} = g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^{1} g_{2,TM}^{1*}] = \frac{1}{2} g_1 g_2^* \tag{127}$$

$$G^0 = \frac{1}{3} g_{1,TM}^{0} g_{2,TM}^{0*} = 0 \tag{128}$$

We then observe that the G^0-term does not contribute to any optical force in the present case of axisymmetric beams of the first kind. Conversely $\mathrm{Re}(G^{11}) = |g_1|^2/2$, proportional to $(S_z)_P$ is at the origin of the scattering forces discussed in the present subsection. The contribution due to the term G^{12} will be discussed later.

5.2 Gradient forces.

In principle, gradient forces are proportional to the gradient of $|\mathbf{E}|^2$ taken at P, e.g. [21] and references therein. We may avoid the calculation of this gradient by noting that it has already been established [21] that $C_{pr,z}^{\mathrm{Im}}$ of Eq.107 for the weak confinement limit of a Gaussian beam described using MLA is a gradient force. Therefore all $C_{pr,z}^{\mathrm{Im}}$ exhibited above, i.e. Eqs.89, 99, 114, and 124 corresponding to the weak confinement of LA, to LA without the weak confinement assumption, to MLA without the weak confinement assumption, and to ISBD respectively, which are all equal to $C_{pr,z}^{\mathrm{Im}}$ of Eq.107, are gradient forces as well. We observe that these gradient forces are proportional to the third power of α. Let us also remark that these gradient forces have been evaluated using Eq.77 which contains coupling terms $g_1 g_2^*$ which imply an interaction between partial waves of order $n = 1$ and partial waves of order $n = 2$.

5.3 Non-standard forces.

Such couplings do not occur in the case of scattering forces because they are proportional to $(S_z)_P = |g_1|^2$ which only involves contributions associated with $n = 1$. These scattering forces are proportional as well to the sixth power of α. However, we also observe other forces which are still proportional to the sixth power of α, but are not proportional to $(S_z)_P = |g_1|^2$. They therefore do not unambiguously deserve to be called scattering forces. They are somehow of a different "nature" insofar as they exhibit couplings between $(n = 1)$- and $(n = 2)$-partial waves.

The origin of all these coupling terms may be already detected in Eq.7 in terms involving $g_{n,TM}^{m} g_{n+1,TM}^{m*}$ and $g_{n,TE}^{m} g_{n+1,TE}^{m*}$, although only the $(n = 1)$-Mie coefficient is retained in the summation. These forces, proportional to the sixth power of α, and which are not scattering forces, have been named axicon forces in the study of circularly symmetric Bessel beams because their existence and manifestations were intimately related to the existence of axicon angles in the beam description, later on non-standard forces when it has been recognized that they were not specific of beams exhibiting axicon terms, as we have done in the present paper up to now. It is furthermore remarkable that they do not appear in the case of the weak confinement limit studied by Lock [21]. Otherwise, they would have already become a matter of debate in 2004.

It is only recently that it has been recognized that these forces, resulting from the Rayleigh limit of GLMT, have their counterparts in the dipole theory of forces. This has been the consequence of two facts (i) that it has been numerically demonstrated in the case of circularly symmetric Bessel beams that the Rayleigh limit of GLMT and the dipole theory of forces numerically agree up to 1,000 decimal places [46] and (ii) that there is a formal identification between both approaches, independently of the kind of beams under consideration [47], these statements holding, for the time being, for longitudinal forces (a similar treatment is under way for transverse forces). Elaborating on Eq.(7) of Chaumet and Nieto-Vesperinas [48], it has been established [47] that the longitudinal optical forces, in both approaches, can be written as:

$$F_z^s = \frac{4\pi}{\eta}\omega\mu\alpha_I \operatorname{Re} S_z + \frac{2\pi}{\eta}\alpha_I \operatorname{Im}[\partial_z |\mathbf{E}|^2] - \frac{2\pi}{\eta}\alpha_I \operatorname{Im}[(\mathbf{E}^*.\boldsymbol{\nabla})E_z] \qquad (129)$$

which is to be evaluated at P, and in which μ is the permeability of the medium, and $\eta = |E_0|^2/2$. Also, α_I is expressed versus the Mie coefficient a_1 according to $\alpha_I = 3\operatorname{Re}(a_1)/(2k^3)$. The prefactors used to write Eq.129 satisfy two conditions (i) that the normalization condition $E_0H_0^*/2$ of GLMT is implemented in the dipole of forces as well and (ii) that F_z^s has been given the dimension of area and can then directly be compared with the cross-sections of GLMT.

Therefore, in Eq.129, the first term is related to the scattering forces of subsection 5.1. Concerning the second term, although depending on $[\partial_z |\mathbf{E}|^2]$, it is not the gradient term of subsection 5.2. Indeed, first, being proportional to α_I, it is proportional to the sixth power of the size parameter, while the gradient term is proportional to the third power. Second, the gradient term involves a real part instead of the imaginary part of the second term of Eq.129. And, finally, because $|\mathbf{E}|^2$ is real, this second term is zero as already mentioned elsewhere, e.g. [46], although we preferred to formally preserve it. The third term is related to non-standard forces (the expression "related to" is to be taken very seriously as we shall explain and demonstrate below). Concerning this third term, a paper by Albaladejo *et al.* in 2009 [49], later commented in 2013, i.e. 4 years later [50], [51], has given to it the name of curl forces. It is maybe a matter of taste whether these curl forces must be called scattering forces as well. Our recommendation however is to give to them a particular name to avoid any confusion. In a review paper published the same year (in 2013) by Marago *et al.*

[52], they received the name of polarization gradient forces. It happens however that these curl forces do not identify yet with the non scattering forces. Indeed, it has been demonstrated elsewhere [47] that they may be expressed in terms of BSCs according to:

$$\begin{aligned} F^c &= -4\pi\alpha_I \,\mathrm{Im}[(\mathbf{E}^*.\boldsymbol{\nabla})E_z] \qquad (130)\\ &= 4\pi\alpha_I \,\mathrm{Im}\{\frac{ik}{3} g^0_{2,TM} g^{0*}_{1,TM} \\ &\quad +ik[(g^1_{2,TM} g^{1*}_{1,TM} + g^{-1}_{2,TM} g^{-1*}_{1,TM}) \\ &\quad -i(g^1_{1,TE} g^{1*}_{1,TM} - g^{-1}_{1,TE} g^{-1*}_{1,TM})]\} \end{aligned}$$

which, under the conditions of Eq.1, reduces to:

$$F^c = 2\pi k\alpha_I [\mathrm{Re}(g_1 g_2^*) - \mathrm{Re}(|g_1|^2)] \qquad (131)$$

The curl forces are therefore the summation of two contributions, the first one corresponding to what we have called non-standard forces, e.g. the contribution due to G^{12} of Eq.127 and the second one, being proportional to the Poynting vector component $(S_z)_P = |g_1|^2$, is a scattering force. In summary, the dipole theory of forces expressed the forces as a summation of (i) scattering forces (ii) gradient forces and (iii) curl forces, e.g. first, second and third terms of Eq.129 respectively while the Rayleigh limit of GLMT expresses the force as a summation of (i) scattering forces (ii) gradient forces and (iii) non-standard forces, with however the proviso that the scattering forces of the dipole theory of forces do not exactly identify with the scattering forces of the Rayleigh limit of GLMT insofar as the curl forces of the former contain a contribution which pertains to the scattering forces of the latter.

It is to be noted that curl forces have been largely ignored for many years. It does not appear in Chaumet and Vesperinas in 2000 [48] and, in 2009, Albaladejo *et al.* [49] commented that "it is usually neglected in the discussion of optical forces on small particles". The identification of non-standard forces is still more recent. They have been actually fortuitously isolated under the name of axicon forces when studying the Rayleigh limit of GLMT [13]. They present the advantage of being a pure coupling term in contrast with curl forces which incorporate as well a scattering force contribution which does not involve any coupling between partial waves of different order. In other words, the curl forces have the disadvantage of summing up two kinds of contribution which, somehow, are of a different nature.

In the present case of Gaussian beams studied in this paper, *and* in the framework of the Rayleigh limit of GLMT *together* with the terminology used in it, we observed that non-standard forces are smaller than scattering forces by an $O(s^2)$-factor in MLA and ISBD, typically 10^{-6} for a typical focussing of the beam and still about 0.026 for a tightly focused beam. They are completely washed out for $s = 0$, i.e. for a plane wave and, more generally, as can be seen from Eq.129 whenever $E_z = 0$, i.e. for any pure TEM waves, in particular for axicon forces of circularly symmetric Bessel beams when the axicon angle is 0, e.g. [13], [22]. In contrast, for $s = 0$ in the LA-case, the NS-forces do not vanish, strictly speaking, but converge to usual scattering forces. This is in agreement with the fact that there is no NS-forces in the weak confinement limit, i.e. for $s = 0$ and even for s small enough, the behavior of LA-Gaussian beams, from the point of view of the existence of $NS-$forces is the same than the one of $TEM-$waves. The case $s = 0$ requires us to make a final last remark, noting that Eq.131 can be rewritten as:

$$F^c = 2\pi k \alpha_I \,\mathrm{Re}[g_1^*(g_2 - g_1)] \tag{132}$$

This equation immediately implies that curl forces vanish when $s = 0$ in all cases, i.e. in LA (see Eqs.81-82), MLA (see Eqs.101-102) and in ISBD (see Eqs.117-118) in agreement with the fact that they have to vanish for pure TEM waves.

6 Conclusion.

This paper discussed the Rayleigh limit of the GLMT in the case of (non dark) on-axis axisymmetric beams of the first kind, with the examples of various representations of Gaussian beams. It provided an opportunity to emphasize the identification between the Rayleigh limit of GLMT and the dipole theory of forces, at least for longitudinal forces (a current work is expected to be able to establish the same kind of identification as far as transverse forces are concerned). The optical forces for point-like (and small enough) particles may then be decomposed, in both formalisms, as a summation of scattering, gradient, and non-standard forces. However, the dipole theory of forces, as can be viewed from Eq.129 is expressed in terms of the total field $\mathbf{E}$, i.e. including all partial waves. It may be argued that actually, insofar as the forces are evaluated at P, the total field will not actually play an effective role, but only the values of the fields at P and the values of the derivatives of the field at P, which is equivalent to retaining only two terms in a Taylor expansion of the field (J.A. Lock, private communication). However, this restriction, which is hidden in the dipole theory of forces, is completely exhibited in the Rayleigh limit of GLMT which expresses the optical forces by retaining only partial waves of order $(n = 1)$ and $(n = 2)$. In this latter formalism, the optical forces are therefore explicitly expressed in terms of low-order BSCs.

The fact that only low-order BSCs intervene in the optical forces on Rayleigh particles may be furthermore viewed as a consequence of the Van de Hulst principle of localization which, by the way, is at the origin of various localized approximations such as the ones used in the present paper, see [41] for a review, [53], [54] for complements, and [55], [56], [57], and references therein, for warnings against the use of localized approximations in the case of beams exhibiting an axicon angle and/or some amount of helicity.

According to this Van de Hulst principle of localization [28], p. 208, a partial wave of order n is associated with rays passing at a distance from the axis equal to $(n+1/2)\lambda/(2\pi)$. Therefore, partial waves of order $n = 1$ and $n = 2$ correspond to rays which pass at distances equal to $3\lambda/(4\pi) \approx \lambda/4$ and $5\lambda/(4\pi) \approx \lambda/2$ from the axis respectively, i.e. at distances significantly smaller than the wavelength. We may loosely state that partial waves of order $n > 2$ pass at distances "too far away" from the axis to be able to interact with the "point-like" particle located on it. A more refined discussion would admit that the influence of each partial wave, depending on the radial dependence of a spherical Bessel function, actually extends from the origin to infinity. But the interaction of the partial wave with the scatterer is strongest where the numerical value of the partial wave is the largest. The location of this largest interaction is summarized in the principle of localization which is convenient to comment as we have done above.

Acknowledgments/Funding.

"National Council for Scientific and Technological Development (CNPq) (426990/2018-8, 307898/2018-0)"

References

[1] G. Gouesbet, B. Maheu, and G. Gréhan. Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation. *Journal of the Optical Society of America A*, 5,9:1427–1443, 1988.

[2] G. Gouesbet, G. Gréhan, and B. Maheu. *Combustion measurements, edited by N. Chigier*, chapter : Generalized Lorenz-Mie theory and applications to optical sizing, pages 339–384. Hemisphere Publishing Corporation, New-York, USA, 1991.

[3] G. Gouesbet and G. Gréhan. *Generalized Lorenz-Mie theories, second edition*. Springer International Publishing AG, 2017.

[4] G. Gouesbet. T-matrix methods for electromagnetic structured beams: A commented reference database for the period 2014-2018. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 230:247–281, 2019.

[5] G. Gouesbet. Van de Hulst Essay: A review on generalized Lorenz-Mie theories with wow stories and epistemological discussion. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 253:Paper 107117, 35 pages, 2020.

[6] F. Corbin, G. Gréhan, and G. Gouesbet. Interaction between a sphere and a Gaussian beam: computations on a micro-computer. *Journal of Particle and Particle Systems Characterization*, 5,3:103–108, 1988.

[7] K.F. Ren, G. Gréhan, and G. Gouesbet. Radiation pressure forces exerted on a particle arbitrarily located in a Gaussian beam by using the generalized Lorenz-Mie theory, and associated resonance effects. *Optics Communications*, 108, 4-6:343–354, 1994.

[8] K.F. Ren, G. Gréhan, and G. Gouesbet. Prediction of reverse radiation pressure by generalized Lorenz-Mie theory. *Applied Optics*, 35,15:2702–2710, 1996.

[9] H. Polaert, G. Gréhan, and G. Gouesbet. Forces and torques exerted on a multilayered spherical particle by a focused Gaussian beam. *Optics Communications*, 155, 1-3:169–179, 1998.

[10] F. Xu, K.F. Ren, G. Gouesbet, X. Cai, and G. Gréhan. Theoretical prediction of radiation pressure force exerted on a spheroid by an arbitrarily shaped beam. *Physical Review E*, 75, Art 026613:1–14, 2007.

[11] F. Xu, J.A. Lock, G. Gouesbet, and C. Tropea. Radiation torque exerted on a spheroid: analytical solution. *Physical Review A*, 78, Art 013843:1–17, 2008.

[12] G. Gouesbet. Generalized Lorenz-Mie theories and mechanical effects of laser light, on the occasion of Arthur Ashkin's receipt ot the 2018 Nobel prize in physics for his pioneering work in optical levitation and manipulation: A review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 225:258–277, 2019.

[13] G. Gouesbet. Gradient, scattering and other kinds of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative and Spectroscopy Transfer*, 246:Article 106913, 7 pages, 2020.

[14] G. Gouesbet. T-matrix formulation and generalized Lorenz-Mie theories in spherical coordinates. *Optics Communications*, 283, 4:517–521, 2010.

[15] J.J. Wang, T. Wriedt, J.A. Lock, and L. Mädler. General description of circularly symmetric Bessel beams of arbitrary order. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 184:218–232, 2016.

[16] J.J. Wang, T. Wriedt, J.A. Lock, and Y.C. Jiao. General description of transverse mode Bessel beams and construction of basis Bessel fields. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 195:8–17, 2017.

[17] J.J. Wang, T. Wriedt, L.Mädler, Y.P. Han, and P. Hartmann. Multipole expansion of circularly Bessel beams of arbitrary order for scattering calculations. *Optics Communications*, 387:102–109, 2017.

[18] J.M. Taylor and G.D. Love. Multipole expansion of Bessel and Gaussian beams for Mie scattering calculations. *Journal of the Optical Society of America A*, 26, 2:278–282, 2009.

[19] J. Chen, J. Ng, P. Wang, and Z. Lin. Analytical partial wave expansion of vector Bessel beam and its application to optical binding. *Optics Letters*, 35, 10:1674–1676, 2010.

[20] J. Chen, J. Ng, P. Wang, and Z. Lin. Analytical partial wave expansion of vector Bessel beam and its application to optical binding: erratum. *Optics Letters*, 36, 7:1243, 2011.

[21] J.A. Lock. Calculation of the radiation trapping force for laser tweezers by use of generalized Lorenz-Mie theory. II. On-axis trapping force. *Applied Optics*, 43,12:2545–2554, 2004.

[22] G. Gouesbet and L.A. Ambrosio. Axicon optical forces and other kinds of transverse optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 258:Paper 107356, 9 pages, 2021.

[23] G. Gouesbet. Optical forces exerted by on-axis Bessel beams on Rayleigh particles in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 260:Paper 107471, 2021.

[24] G. Gouesbet. Partial wave expansions and properties of axisymmetric light beams. *Applied Optics*, 35,9:1543–1555, 1996.

[25] G. Gouesbet. Poynting theorem in terms of beam shape coefficients and applications to axisymmetric, dark and non-dark, vortex and non-vortex beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:184–196, 2017.

[26] G. Gouesbet and J.A. Lock. A darkness theorem for the beam shape coefficients and its relationship to higher-order non vortex Bessel beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:229–235, 2017.

[27] G. Gouesbet and G. Gréhan. *Generalized Lorenz-Mie theories*. Springer, Berlin, 2011.

[28] H.C. van de Hulst. *Light scattering by small particles.* Wiley, New York, 1957.

[29] G. Gouesbet and G. Gréhan. Sur la généralisation de la théorie de Lorenz-Mie. *Journal of Optics*, 13,2:97–103, 1982.

[30] L. Robin. *Fonctions sphériques de Legendre et fonctions sphéroidales. Volumes 1, 2, 3.* Gauthier-Villars, Paris, 1957.

[31] G.B. Arfken and H.J. Weber. *Mathematical methods for physicists, sixth edition.* Elsevier Academic Press, Amsterdam, 2005.

[32] L.W. Davis. Theory of electromagnetic beams. *Physical Review*, 19, 3:1177–1179, 1979.

[33] G. Gouesbet, J.A. Lock, and G. Gréhan. Partial wave representations of laser beams for use in light scattering calculations. *Applied Optics*, 34,12:2133–2143, 1995.

[34] J.P. Barton and D.R. Alexander. Fifth-order corrected electromagnetic field components for fundamental Gaussian beams. *Journal of Applied Physics*, 66,7:2800–2802, 1989.

[35] G. Gréhan, B. Maheu, and G. Gouesbet. Scattering of laser beams by Mie scatter centers: numerical results using a localized approximation. *Applied Optics*, 25,19:3539–3548, 1986.

[36] B. Maheu, G. Gréhan, and G. Gouesbet. Generalized Lorenz-Mie theory: first exact values and comparisons with the localized approximation. *Applied Optics*, 26,1:23–25, 1987.

[37] G. Gouesbet, G. Gréhan, and B. Maheu. Computations of the g_n coefficients in the generalized Lorenz-Mie theory using three different methods. *Applied Optics*, 27,23:4874–4883, 1988.

[38] B. Maheu, G. Gréhan, and G. Gouesbet. Ray localization in Gaussian beams. *Optics Communications*, 70,4:259–262, 1989.

[39] K.F. Ren, G. Gréhan, and G. Gouesbet. Evaluation of laser sheet beam shape coefficients in generalized Lorenz-Mie theory by use of a localized approximation. *Journal of the Optical Society of America A*, 11,7:2072–2079, 1994.

[40] J.A. Lock and G. Gouesbet. Rigorous justification of the localized approximation to the beam shape coefficients in generalized Lorenz-Mie. I. On-axis beams. *Journal of the Optical Society of America A*, 11,9:2503–2515, 1994.

[41] G. Gouesbet, J.A. Lock, and G. Gréhan. Generalized Lorenz-Mie theories and description of electromagnetic arbitrary shaped beams: localized approximations and localized beam models, a review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 112:1–27, 2011.

[42] G. Gouesbet and J.A. Lock. Rigorous justification of the localized approximation to the beam shape coefficients in generalized Lorenz-Mie theory. II. Off-axis beams. *Journal of the Optical Society of America A*, 11,9:2516–2525, 1994.

[43] G. Gouesbet. Exact description of arbitrary shaped beams for use in light scattering theories. *Journal of the Optical Society of America A*, 13,12:2434–2440, 1996.

[44] G. Gouesbet. Higher-order descriptions of Gaussian beams. *Journal of Optics (Paris)*, 27,1:35–50, 1996.

[45] H. Polaert, G. Gréhan, and G. Gouesbet. Improved standard beams with applications to reverse radiation pressure. *Applied Optics*, 37,12:2435–2440, 1998.

[46] L.A. Ambrosio and G. Gouesbet. On longitudinal radiation pressure cross-section in the generalized Lorenz-Mie theory and its numerical relationship with the dipole theory of forces. *Journal of Optica Society of America B. In Press.*

[47] L.A. Ambrosio and G. Gouesbet. On the Rayleigh limit of the generalized Lorenz-Mie theory and its formal identification with the dipole theory of forces. I. The longitudinal case. *Journal of Quantitative Spectroscopy and Radiative Transfer. In Press.*

[48] P.C. Chaumet and M. Nieto-Vesperinas. Time-averaged total force on a dipolar sphere in an electromagnetic field. *Optics Letters*, 25, 15:1065–1067, 2000.

[49] S. Albaladejo, M.I. Marqués, M. Laroche, and J.S. Saenz. Scattering forces from the curl of the spin angular momentum of a light field. *Physical Review Letters*, 102:Article 113602, 2009.

[50] D.B. Ruffner and D.G. Grier. Comment on "Scattering forces from the curl of the spin angular momentum of a light field"Í. *Physical Review Letters*, 111, 059301:1 page, 2013.

[51] M.I. Marques and J.J. Saenz. Marques and Saenz Reply. *Physical Review Letters*, 111, 059302:1 page, 2013.

[52] O.N. Marago, P.H. Jones, P.G. Gucciardi, G. Volpe, and A.C Ferrari. Optical trapping and manipulation of nanostructures. *Nature nanotechnology*, 8:807–819, 2013.

[53] J.J. Wang and G. Gouesbet. Note on the use of localized beam models for light scattering theories in spherical coordinates. *Applied Optics*, 51, 17:3832–3836, 2012.

[54] G. Gouesbet. Second modified localized approximation for use in generalized Lorenz-Mie theories and other theories revisited. *Journal of the Optical Society of America A*, 30, 4:560–564, 2013.

[55] G. Gouesbet, L.A. Ambrosio, and L.F.M. Votto. Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam focused by a lens in an on-axis configuration. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 242:Paper 106759, 17 pages, 2019.

[56] G. Gouesbet and L.A. Ambrosio. On the validity of the use of a localized approximation for helical beams. I. Formal aspects. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 208:12–18, 2018.

[57] L.A. Ambrosio and G. Gouesbet. On the validity of the use of a localized approximation for helical beams. II. Numerical aspects. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 215:41–50, 2018.