Local Linear Convergence of Inertial Forward-Backward Splitting for Low Complexity Regularization
Jingwei Liang, Jalal M. Fadili, Gabriel Peyré

To cite this version:
Jingwei Liang, Jalal M. Fadili, Gabriel Peyré. Local Linear Convergence of Inertial Forward-Backward Splitting for Low Complexity Regularization. SPARS, 2015, Cambridge, France. hal-02456434

HAL Id: hal-02456434
https://hal-normandie-univ.archives-ouvertes.fr/hal-02456434
Submitted on 27 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Local Linear Convergence of Inertial Forward–Backward Splitting for Low Complexity Regularization

Jingwei Liang and Jalal M. Fadili
CNRS, GREYC, ENSICAEN, Université de Caen
Email: Jingwei.Liang, Jalal.Fadili)@ensicaen.fr

Gabriel Peyré
CNRS, Cermade, Université Paris-Dauphine
Email: Gabriel.Peyre@ceremade.dauphine.fr

Abstract—In this abstract, we consider the inertial Forward-Backward (iFB) splitting method and its special cases (Forward-Backward/ISTA and FISTA). Under the assumption that the non-empty convex part of the objective is partly smooth relative to an active smooth manifold, we show that iFB-type methods (i) identify the active manifold in finite time, then (ii) enter a local linear convergence regime that we characterize precisely. This gives a grounded and unified explanation to the typical behaviour that has been observed numerically for many low-complexity regularizers, including $\ell_1$, $\ell_2,1$-norms, total variation (TV) and nuclear norm to name a few. The obtained results are illustrated by concrete examples.

I. INTRODUCTION
Consider the following structured optimization problem

$$
\min_{x \in \mathbb{R}^n} \{ \Phi(x) \triangleq F(x) + J(x) \},
$$

where $J \in \Gamma_0(\mathbb{R}^n)$, the proper, lower semi-continuous and convex functions, $F$ is convex, $C^{1,1}(\mathbb{R}^n)$ with $\nabla F$ being $\beta$-Lipschitz continuous. We assume that $\text{Argmin} \Phi \neq \emptyset$.

In this paper, we consider a generic form of inertial Forward–Backward for solving (P) which reads,

$$
y_k = x_k + \alpha_k (x_k - x_{k-1}),
\quad
x_{k+1} = \text{Prox}_{\gamma \nabla F} (y_k - \gamma_k \nabla F(y_k)),
$$

where $\alpha_k \in [0, a]$ and $b_k \in [0, b]$, $(a, b) \in [0, 1]^2$, and the step-size $0 < \gamma \leq \gamma_k \leq \gamma < \min(2\alpha\beta^2, 2\beta^2)$. For $\gamma > 0$, the proximity operator is defined as $\text{Prox}_{\gamma \nabla F} (x) = \text{argmin}_{x \in \mathbb{R}^n} \frac{1}{2} \| x - z \|^2 + J(x)$.

iFB (I.1) covers various special cases in the literature, including the (unrelaxed) Forward–Backward (FB) [1] and FISTA [2]. In the original FISTA, only convergence of the objective function is guaranteed. Recently in [5], the iterates are proved to be convergent under $\alpha_k = b_k = (t_k - 1)/t_k$ where $t_k = (k + p - 1)/p, p \geq 2$.

II. PARTLY SMOOTH FUNCTIONS AND FINITE IDENTIFICATION
The class of partly smooth functions [3], is specialized here to functions in $\Gamma_0(\mathbb{R}^n)$. Denote $\text{par}(C)$ the linear subspace parallel to the non-empty convex set $C \subset \mathbb{R}^n$, and $\text{ri}(C)$ its relative interior.

Definition II.1. Let $J \in \Gamma_0(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ such that $\partial J(x) \neq \emptyset$. $J$ is partly smooth at $x$ relative to a set $M$ containing $x$ if

- (Smoothness) $M$ is a $C^2$-manifold, $J|_M$ is $C^2$ around $x$;
- (Sharpness) The tangent space $T_M(x) = T_xM \triangleq \text{par}(\partial J(x))^\perp$;
- (Continuity) $\partial J$ is continuous at $x$ relative to $M$.

Examples of such functions are given in Section IV, see also [4].

Theorem II.2 (Finite activity identification). Suppose $x^k$ converges to a minimizer $x^*$ of (P) such that $J$ is partly smooth at $x^*$ relative to $M_{x^*}$, and

$$
- \nabla F(x^*) \in \text{ri}(\partial J(x^*)),
$$

then there exists a $K > 0$ such that for all $k \geq K$, $x_k \in M_{x^*}$. If moreover $M_{x^*}$ is affine/linear, then $y_k, x_k \in M_{x^*}$, for $k > K$.

Condition (II.1) can be viewed as a geometric generalization of the strict complementarity of non-linear programming, and is almost necessary for the finite identification of $M_{x^*}$ [3].

III. LOCAL LINEAR CONVERGENCE
We now turn to the local linear convergence of the iFB-type methods with partly smooth functions. For space limitations, we mainly focus on the case where $a_k = b_k$, and denote $d_k^{k+1} = \left( x_k^{k+1} - x^* \right)$.

Theorem III.1. We assume the conditions of Theorem II.2 hold. If moreover $F$ is $C^2$ near $x^*$ and there exists $\alpha \geq 0$ such that $\mathbb{P}_{T_{x^*}} \nabla^2 F(x^*) \preceq \alpha I$. Then for all $k$ large enough, we have

1) $Q$-linear rate: if $0 < \gamma \leq \gamma_k \leq \gamma < \min(2\alpha\beta^2, 2\beta^2)$, then there exists a $\rho_k$, the iterates satisfy

$$
\| x_k^{k+1} - x^* \| \leq \rho_k \| d_k^k \|, \\
\text{where } \eta = \max \{ q(\gamma), q(\eta) \} \in [0, 1], q(\gamma) = 1 - 2\alpha\gamma + \beta^2 \gamma^2,
$$

2) $R$-linear rate: if $M_{x^*}$ is affine/linear, then

$$
\| x_k^{k+1} - x^* \| \leq \rho_k \| d_k^k \|,
$$

where $\rho_k \in [0, 1]$

$$
\rho_k = \begin{cases}
\frac{1}{\sqrt{\eta_k}}, & \eta_k \in (-1, 0) \cup \left( \frac{4\alpha_k}{(1+a_k)^2}, 1 \right], \\
\frac{4\alpha_k}{(1+a_k)^2}, & \eta_k \in [0, \frac{4\alpha_k}{(1+a_k)^2}]
\end{cases}
$$

and $\eta_k \in [-1, 1]$ is an eigenvalue of $Id - \gamma_k P_{T_{x^*}}$, $\int_{T_{x^*}} \nabla^2 F(x^*) + t(y_k - x^*) dt P_{T_{x^*}}$.

IV. NUMERICAL EXPERIMENTS

Example IV.1 ($\ell_1$-norm). The $\ell_1$-norm is partly smooth relative to $M = T_2 = \{ u \in \mathbb{R}^n : \text{supp}(u) \subseteq \text{supp}(x) \}$.

Example IV.2 ($\ell_1, 2$-norm). $\ell_1,2$-norm is partly smooth relative to $M = T_2 = \{ u \in \mathbb{R}^n : \text{supp}_2(u) \subseteq \text{supp}_2(x) \}$, where $\text{supp}_2(x) = \bigcup \{ b : x_b \neq 0 \}$ and $\bigcup_{b \in B} b = \{ 1, \ldots, n \}$.

Example IV.3 (TV semi-norm). The TV semi-norm $\| x \|_{TV} = \| \nabla x \|$, is partly smooth relative the subspace $M = T_2 = \{ u \in \mathbb{R}^n : \text{supp}(\nabla u) \subseteq I \}$, $I = \text{supp}(\nabla x)$.

Example IV.4 (Nuclear norm). The nuclear norm is partly smooth relative to the manifold of fixed rank matrices, $M = \{ z \in \mathbb{R}^{n_1 \times n_2} : \text{rank}(z) = r \}$.

We now consider the problem $\min_{x \in \mathbb{R}^n} \frac{1}{2} \| y - Ax \|_2^2 + \lambda J(x)$, where $y \in \mathbb{R}^m$ is the observation, $A : \mathbb{R}^m \to \mathbb{R}^n$ is drawn from the standard Gaussian ensemble, and $\lambda > 0$ is the regularization parameter. The convergence profiles are depicted in Figure 1.
Fig. 1: Local linear convergence of IFB-type methods in terms of $\|x^k - x^*\|$. The forward model of the problem of interests reads $y = Ax_0 + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \delta^2)$. (a) $\ell_1$-norm, $(m, n) = (48, 128)$, $x_0$ is 8-sparse; (b) $\ell_{1,2}$-norm, $(m, n) = (60, 128)$, $x_0$ has 3 non-zero blocks with block-size 4; (c) 1D TV semi-norm, $(m, n) = (48, 128)$, $\nabla x_0$ is 8-sparse; (d) Nuclear norm, $(m, n) = (1425, 2500)$, $x_0 \in \mathbb{R}^{50 \times 50}$ and $\text{rank}(x_0) = 5$. The red, black and blue lines are respectively the results of FB, FISTA [5] and iFB (with $a_k = b_k \equiv \sqrt{5} - 2.01$).

All algorithms were tested with $\gamma_k \equiv 1/\|A\|_2$. The solid lines are the practical observed profiles and the dashed ones the theoretical predictions. The beginning of the dashed lines are the points when $x^k$ identifies the manifold $\mathcal{M}_{x^{*}}$. As one can observe, FISTA has the fastest manifold identification, however, locally it is the slowest for all tested examples. Indeed, when the manifold is affine, it can be shown from Theorem III.1 that $\rho_k \in [\eta_k, \sqrt{\eta_k}]$ for $a_k > \eta_k$, i.e. FISTA is locally slower than FB.

ACKNOWLEDGMENT

This work has been partly supported by the European Research Council (ERC project SIGMA-Vision). JF was partly supported by Institut Universitaire de France.

REFERENCES