Reconstruction of unknown storativity and transmissivity functions in 2D groundwater equations

Adel Hamdi, Abderrahim Jardani

Reconstruction of unknown storativity and transmissivity functions in $2D$ groundwater equations

Adel Hamdi*[1] and Abderrahim Jardani†[2]

[1]Laboratoire de Mathématiques LMI, INSA Rouen Normandie, 76801 Saint-Etienne-du-Rouvray, France

[2]Laboratoire de M2C, Université de Rouen, Mont Saint Aignan, France

Abstract

The paper deals with the identification of unknown storativity and transmissivity distributions within a $2D$ confined aquifer using pumping tests. We introduce a change of variables that transforms the groundwater equation into a diffusion-reaction one, where the diffusion term is the fraction transmissivity/storativity whereas the reaction term yields the right hand side of a second order nonlinear partial differential equation satisfied by the unknown storativity function. Using records of the drawdown at some measuring wells within the monitored aquifer, we establish identifiability results on the introduced diffusion and reaction terms as well as on the storativity values at the employed wells. We develop an identification approach that starts by determining the auxiliary diffusion and reaction variables. Afterwards, this approach uses an assumption related to the incompressibility of water to develop a local determination procedure of the unknown storativity function. Besides, based on the interpolation of its values at the employed wells, a global determination procedure of this function is also developed. The unknown transmissivity is then determined by the product of the identified storativity and fraction transmissivity/storativity functions. Some numerical experiments are presented.

1 Introduction

Managing effectively groundwater resources requires the knowledge of some hydraulic properties defining the nature of the involved aquifers. For instance, among the main required properties we quote the following, see [21, 32] *Storativity:* That is the volume of water released from storage with respect to the change in head (water level) and to the aquifer's surface area. *Transmissivity:* It represents the aquifer's ability to transmit groundwater throughout its entire saturated thickness. Since those properties affect significantly groundwater movement and storage as well as solute transport, in the literature numerous studies have been devoted to estimate such properties. Those studies

*adel.hamdi@insa-rouen.fr

†abderrahim.jardani@univ-rouen.fr

are mainly based on the so-called pumping test analysis [7, 18, 21, 34] which consists of analysing data, measured in some surrounding well tests, that represents the aquifer's response to a hydraulic forcing term introduced through one or multi-pumping wells. In practice, accurate estimations of the hydraulic properties lead to employ more appropriate actions in large spectrum of applications that go from groundwater exploration to waste-disposal evaluation as well as to the determination of efficiency and productivity of a well for groundwater extraction. For example, estimating transmissivity could help well field managers to design more energy-efficient pumping schemes since Sterrett reported in [29] that the energy requirement for pumping is directly proportional to the hydraulic lift. In addition, an accurate estimation of storativity is important for quantifying groundwater availability in order to satisfy drinking water demand, for instance, municipal wells in Colorado extract over 100 million cubic meters of groundwater per year.

In the literature, the identification of aquifers hydraulic properties has been initiated by Theis in [33] who used the so-called type-curve matching method to estimate aquifers parameters. Then, Cooper and Jacob employed in [11] the straight-line method to determine hydraulic properties in groundwater equations. Those techniques, called also graphical approach, are based on matching graphically the recorded data at the pumping tests to the simulations of several analytical models depending on the type of aquifers and the hydraulic conditions. In the last few decades when the use of computer became widely available, we have seen emerging several new approaches extending the graphical approach to solve applications with big data fitting and to estimate wider range of hydraulic properties as well as to explore larger class of aquifers. Those approches employ different techniques to identify the underlined hydraulic properties, for instance, stochastic techniques have been employed in [16] and genetic algorithms in [1] whereas geostatistical inversion of data by using the Bayesian approach have been used in [5, 8, 10, 18] and probabilistic estiamtions using time series model in [28]. Besides, there is a direct identification approach developed in [22, 24] that consists of considering the transient groundwater equation along the streamlines associated to the gradient of the drawdown as a first order ordinary differential equation of the unknown transmissivity whereas the storativity and the drawdown are both supposed to be known. Provided an initial value of the transmissivity is given for each streamline, this approach transforms the identification of the transmissivity into solving a Cauchy problem. However, the knowledge of the streamlines and of an initial transmissivity value for each streamline are difficult to achieve in the implementation of a real case. In [25], the authors introduced the so-called Differential System (DS) method that, given the drawdown for three different flows, solves the Cauchy problem in a way that doesn't require anymore the knowledge of streamlines, an initial transmissivity value is needed in only one point and no a priori knowledge of the storativity is required. Nevertheless, some remarks have been made by Beckie in [5] that the estimation of storativity is a very sensitive problem since it is influenced by the estimated transmissivity. Meier in [21] reported that the estimation of those properties in heterogeneous media depends on the measurement locations.

In the present study, we focus on identifying unknown storativity and transmissivity distributions within a $2D$ confined aquifer using pumping tests. Based on the analysis and optimisation of the groundwater partial differential equation governing the drawdown in the considered aquifer, we develop an identification approach that uncouples the determination of the unknown storativity function from the identification of the unknown transmissivity. Moreover, the developed approach establishes conditions on the used pumping source as well as on the number and the locations of the employed measuring wells to ensure uniqueness of the involved unknown functions. The paper is organized as follows: Section 2 is devoted to introduce the problem statement and to establish some technical results for later use. In section 3, we study the identifiability of the occurring unknown functions. Section 4, is reserved to develop the identification approach that leads to determine the unknown storativity and transmissivity functions. Some numerical experiments on a variant of groundwater equations are presented in section 5.

2 Problem statement and technical results

Let $T > 0$ be a finite final monitoring time and Ω be a bounded and connected open subset of $I\!R^2$ with Lipschitz boundary $\partial\Omega$. According to [3, 4, 11, 32, 33], it follows that due to the similarity between groundwater flow and heat conduction, the hydraulic head (water level or also called drawdown), denoted here by u, in a confined aquifer Ω subject to an external hydraulic pumping source f is governed by the following system:

$$\begin{cases} L[u](x_1, x_2, t) = f(x_1, x_2, t) & \text{in } \Omega \times (0, T) \\ u(\cdot, 0) = 0 & \text{in } \Omega \\ P\nabla u \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{1}$$

where ν is the unit outward vector normal to $\partial\Omega$ and L is the second order linear partial differential operator defined by

$$L[u](x_1, x_2, t) := S(x_1, x_2)\partial_t u(x_1, x_2, t) - \operatorname{div}\big(P(x_1, x_2)\nabla u(x_1, x_2, t)\big) \tag{2}$$

In (2), S and P designate the storativity and the transmissivity functions defining the nature of the understudy aquifer Ω. In the remainder of this paper, we consider that S and P are two differentiable functions that belong to the following admissible set:

$$\mathcal{A} := \big\{0 < P \in W^{1,\infty}(\Omega),\ 0 < S \in W^{2,\infty}(\Omega) \quad \text{and} \quad P\nabla S \cdot \nu = 0 \ \text{ on } \partial\Omega\big\} \tag{3}$$

We employ a single pumping well that reaches the aquifer at the point $a \in \Omega$ through which a hydraulic time-dependent forcing function $\ell \in L^2(0, T)$ is pumped. Therefore, the external time-dependent pumping source f involved in (1) is defined by

$$f(x_1, x_2, t) = \ell(t)\delta_a(x_1, x_2), \quad \text{for all } (x_1, x_2, t) \in \Omega \times (0, T) \tag{4}$$

where δ_a denotes the Dirac mass at the pumping position a. Thus, given S and P elements of the set $\mathcal{A}$ together with $a \in \Omega$ and $\ell \in L^2(0, T)$ defining f in (4), the *forward problem* (1)-(4) admits a unique solution u that belongs to the functional space, see [20, 27]:

$$L^2\big(0, T; L^2(\Omega)\big) \cap \mathcal{C}^0\big(0, T; H^{-1}(\Omega)\big) \tag{5}$$

Moreover, the state u is sufficiently regular in $\Omega \setminus \{a\}$. Therefore, given the positions of some measuring wells $b^{i=1,\dots,I} \in \Omega \setminus \{a\}$, we define the observation operator as follows:

$$M[S, P] := \big\{u(b^i, t) \ \text{ for all } t \in (0, T), \ \text{ for } i = 1, \dots, I\big\} \tag{6}$$

Later on in this paper, the number $I \in I\!N^*$ of measuring wells and their positions $b^{i=1,\dots,I}$ with respect to the pumping location $a \in \Omega$ will be further discussed.

The *nonlinear inverse problem* with which we are concerned here consists of: Given time records $d_i(t)$, $\forall t \in (0,T)$ of the state u taken at the measuring wells $b^{i=1,\dots,I}$, determine the two unknown functions S and P of $\mathcal{A}$ involved in the problem (1)-(4) that yield

$$M[S,P] = \big\{d_i(t) \ \text{ for all } t \in (0,T), \quad \text{for } i = 1,\dots,I\big\} \tag{7}$$

For later use, to each two differentiable functions S and P of the admissible set $\mathcal{A}$, we associate the intermediate variables ψ, Ψ and ρ defined as follows:

$$\psi = \frac{P}{S}, \qquad \Psi = \frac{1}{S}\nabla S \qquad \text{and} \qquad \rho = \frac{1}{4}\psi\|\Psi\|_2^2 + \frac{1}{2}\mathrm{div}\big(\psi\Psi\big) \tag{8}$$

where $\|\cdot\|_2$ denotes the euclidean norm. Notice that in the case when the two functions ψ and ρ are both known, it follows from (8) that the unknown storativity function S solves in Ω the following second order nonlinear partial differential equation:

$$\frac{\psi}{4}\|\Psi\|_2^2 + \frac{1}{2}\nabla\psi\cdot\Psi + \frac{\psi}{2}\mathrm{div}\big(\Psi\big) = \rho \quad \Leftrightarrow \quad \frac{1}{S}\Big(\Delta S + \big[\frac{1}{\psi}\nabla\psi - \frac{1}{2S}\nabla S\big]\cdot\nabla S\Big) = \frac{2\rho}{\psi} \tag{9}$$

Besides, using the auxiliary variables ψ and ρ introduced in (8), we consider the eigenvalue problem: For all $n \in I\!N$, find μ_n and ξ_n that solve the system:

$$\begin{cases} -\mathrm{div}\big(\psi\nabla\xi_n\big) + \rho\xi_n = \mu_n\xi_n & \text{in } \Omega \\ \psi\nabla\xi_n\cdot\nu = 0 \quad \text{on } \partial\Omega \end{cases} \tag{10}$$

For the simplicity of our notations, in the remainder we denote by $\{\xi_n\}$ the set of normalized eigenfunctions solutions of (10). Then, according to [27], we have:

Theorem 2.1 *(See [27]) Let Ω be a bounded open subset of $I\!R^2$ with Lipschitz boundary $\partial\Omega$. The normalized eigenfunctions $\{\xi_n\}$ solutions of the system (10) form a complete orthonormal family of $L^2(\Omega)$ and their associated eigenvalues (μ_n) form an increasing sequence of real numbers that tends to infinity.*

In the spectral Neumann decomposition Theorem 2.1, the Lipschitz condition on the boundary $\partial\Omega$ is required for the compactness of $H^1(\Omega)$ imbedding in $L^2(\Omega)$. Moreover, as far as the first eigenpair of the eigenvalue problem (10) is concerned, we quote the following properties: For more details, see for instance [6, 9].

Remark 2.2 *The principal eigenvalue μ_0 of the problem (10) is unique i.e., $\mu_0 < \mu_n$, $\forall n \in I\!N^*$. In addition, for the case when ρ is a real number, the first eigenpair solution of the eigenvalue problem (10) is defined by: $\mu_0 = \rho$ and $\xi_0(x_1,x_2) = 1/\sqrt{\mathcal{S}(\Omega)}$ for all $(x_1,x_2) \in \Omega$, where $\mathcal{S}(\Omega)$ denotes the surface area of the domain Ω.*

Furthermore, we introduce what we will refer to in the remainder as strategic position.

Definition 2.3 *Let $\{\zeta_n\}$ be a complete orthogonal family of continuous functions in $L^2(\Omega)$. We say that $(\hat{x}_1,\hat{x}_2) \in \Omega$ is strategic with respect to $\{\zeta_n\}$ if $\zeta_n(\hat{x}_1,\hat{x}_2) \neq 0, \forall n$.*

The notion of strategic position in the sense of Definition 2.3 is well known in the literature. Indeed, this notion has been introduced by El Jai and Pritchard in [13] and used by many other authors, for example, in [12, 17]. For later use, to each measuring position $b^i \in \Omega \setminus \{a\}$ we associate an impulse response G_{b^i} that solves:

$$\begin{cases} -\mathrm{div}\big(\psi\nabla G_{b^i}\big) + \rho G_{b^i} = \delta_{b^i}(x_1,x_2) \quad \text{in } \Omega \\ \psi\nabla G_{b^i}\cdot\nu = 0 \quad \text{on } \partial\Omega \end{cases} \tag{11}$$

Let G_a be the solution of the system (11) with a instead of b^i. From multiplying the first equation in (11) by G_a and integrating by parts over Ω using Green's formula, we obtain

$$\begin{aligned} G_a(b^i) &= \int_\Omega \psi\nabla G_a\nabla G_{b^i} + \int_\Omega \rho G_a G_{b^i} \\ &= \langle -\mathrm{div}\big(\psi\nabla G_a\big) + \rho G_a, G_{b^i}\rangle \\ &= G_{b^i}(a) \end{aligned} \tag{12}$$

where $\langle , \rangle$ represents the product in the distribution sense. The result in (12) yields a symmetric property of the impulse response solution of the system (11). Moreover, using the complete orthonormal family $\{\xi_n\}$, the solution of (11) is given by

$$G_{b^i}(x_1,x_2) = \sum_{n\geq 0}\frac{\xi_n(b^i)}{\mu_n}\xi_n(x_1,x_2), \quad \forall(x_1,x_2)\in\Omega \tag{13}$$

Besides, we employ the change of variables: $U(x_1,x_2,t) = \sqrt{S(x_1,x_2)}u(x_1,x_2,t)$ in $\Omega\times(0,T)$. That leads to $\nabla u = S^{\frac{1}{2}}\big(\frac{1}{S}\nabla U - \frac{U}{2S^2}\nabla S\big)$. Afterwards, using the intermediate variables ψ, Ψ and ρ introduced in (8), we get

$$\begin{aligned} \mathrm{div}\big(P\nabla u\big) &= \mathrm{div}\big(S^{\frac{1}{2}}\big[\psi\nabla U - \frac{1}{2}\psi\Psi U\big]\big) \\ &= S^{\frac{1}{2}}\Big(\mathrm{div}\big(\psi\nabla U\big) - \big[\tfrac{1}{4}\psi\|\Psi\|_2^2 + \tfrac{1}{2}\mathrm{div}\big(\psi\Psi\big)\big]U\Big) \\ &= S^{\frac{1}{2}}\Big(\mathrm{div}\big(\psi\nabla U\big) - \rho U\Big) \end{aligned} \tag{14}$$

Since in view of (3) we have $P\nabla S\cdot\nu = 0$ on $\partial\Omega$, it follows that $P\nabla u\cdot\nu = S^{\frac{1}{2}}\psi\nabla U\cdot\nu$ on $\partial\Omega$. Hence, the problem (1)-(2) is equivalent to the following system:

$$\begin{cases} \partial_t U - \mathrm{div}\big(\psi\nabla U\big) + \rho U = S^{-\frac{1}{2}}f \quad \text{in } \Omega\times(0,T) \\ U(\cdot,0) = 0 \qquad \text{in } \Omega \\ \psi\nabla U\cdot\nu = 0 \quad \text{on } \partial\Omega\times(0,T) \end{cases} \tag{15}$$

Furthermore, as the set of normalized eigenfunctions $\{\xi_n\}$ solutions of the eigenvalue problem (10) forms a complete orthonormal family of $L^2(\Omega)$, the solution U of the system (15) using the pumping source f given in (4), can be written under the form

$$U(x_1,x_2,t) = \sum_{n\geq 0} e_n(t)\xi_n(x_1,x_2), \qquad \text{where} \quad \begin{cases} e_n'(t) + \mu_n e_n(t) = \frac{\xi_n(a)}{\sqrt{S(a)}}\ell(t), \quad \forall t\in(0,T) \\ e_n(0) = 0 \end{cases} \tag{16}$$

Therefore, from (16) it follows that: For all $(x_1, x_2, t) \in \Omega \times (0, T)$,

$$U(x_1, x_2, t) = \frac{1}{\sqrt{S(a)}} \sum_{n \geq 0} \xi_n(a)\xi_n(x_1, x_2) \int_0^t \ell(\eta) e^{-\mu_n(t-\eta)} d\eta \tag{17}$$

Then, we establish the following technical result that leads to express the value of the unknown storativity function S at the measuring wells $b^{i=1,\ldots,I} \in \Omega \setminus \{a\}$ in terms of its value at the pumping well $a \in \Omega$:

Lemma 2.4 *Let $a \in \Omega$, $\ell \in L^2(0, T)$ and $f(x_1, x_2, t) = \ell(t)\delta_a(x_1, x_2)$ be the pumping source employed in the system (15). For all $b^i \in \Omega \setminus \{a\}$, it holds*

$$\sqrt{S(b^i)} = \frac{1}{\sqrt{S(a)}} B^i \qquad \text{where: } B^i = \frac{\displaystyle\sum_{n \geq 0} \frac{\xi_n(a)\xi_n(b^i)}{\mu_n} \int_0^T \ell(t)\big(1 - e^{-\mu_n(T-t)}\big) dt}{\displaystyle\int_0^T u(b^i, t) dt} \tag{18}$$

Moreover, if the forcing function ℓ leads the solution U of the system (15) to satisfy $U(\cdot, T) = 0$ a.e. in Ω then, the form in (18) of the coefficients B^i is reduced to

$$B^i = \frac{\displaystyle\int_0^T \ell(t) dt}{\displaystyle\int_0^T u(b^i, t) dt} G_{b^i}(a) \tag{19}$$

where G_{b^i} is the impulse response solution of the problem (11).

Proof. From multiplying the first equation of the system (11) by the solution U of the problem (15), where f is the pumping source defined in (4), and integrating by parts over Ω using Green's formula, we get: For all $t \in (0, T)$,

$$\begin{aligned} U(b^i, t) &= \langle -\text{div}(\psi \nabla U) + \rho U, G_{b^i} \rangle \\ &= \frac{G_{b^i}(a)}{\sqrt{S(a)}} \ell(t) - \frac{d}{dt} \langle U, G_{b^i} \rangle_{L^2(\Omega)} \end{aligned} \tag{20}$$

where $\langle , \rangle$ is the product in the distribution sense. Since $U(b^i, t) = \sqrt{S(b^i)} u(b^i, t)$ for all $t \in (0, T)$ and $U(\cdot, 0) = 0$ in Ω, it follows from integrating (20) over $(0, T)$ that

$$\sqrt{S(b^i)} = \frac{\displaystyle\frac{G_{b^i}(a)}{\sqrt{S(a)}} \int_0^T \ell(t) dt - \langle U(\cdot, T), G_{b^i} \rangle_{L^2(\Omega)}}{\displaystyle\int_0^T u(b^i, t) dt} \tag{21}$$

If the time-dependent forcing function $\ell \in L^2(0, T)$ employed in (15) yields $U(\cdot, T) = 0$ a.e. in Ω then, from (21) we find the form of the coefficients B^i announced in (19). Otherwise, from using (17) to compute the final state $U(\cdot,T)$ and by replacing the impulse response G_{b^i} by its value given in (13), we obtain

$$\begin{aligned}\langle U(\cdot,T),G_{b^i}\rangle_{L^2(\Omega)} &= \frac{1}{\sqrt{S(a)}}\Big\langle \sum_{n\geq 0}\xi_n(a)\int_0^T \ell(t)e^{-\mu_n(T-t)}dt\,\xi_n,\sum_{k\geq 0}\frac{\xi_k(b^i)}{\mu_k}\xi_k\Big\rangle_{L^2(\Omega)} \\ &= \frac{1}{\sqrt{S(a)}}\sum_{n\geq 0}\frac{\xi_n(a)\xi_n(b^i)}{\mu_n}\int_0^T \ell(t)e^{-\mu_n(T-t)}dt\end{aligned} \tag{22}$$

The second equality in (22) holds since $\{\xi_n\}$ forms an orthonormal family of $L^2(\Omega)$. Afterwards, replacing in (21) the term $\langle U(\cdot,T),G_{b^i}\rangle_{L^2(\Omega)}$ by its value obtained in (22) and $G_{b^i}(a)$ using (13) leads to the result announced in (18). ■

Remark 2.5 *Since $u(x_1,x_2,t)=U(x_1,x_2,t)/\sqrt{S(x_1,x_2)}$ for all $(x_1,x_2,t)\in\Omega\times(0,T)$, it follows from (17) and according to Lemma 2.4 that: For all $b^i\in\Omega\setminus\{a\}$,*

$$u(b^i,t)=\frac{1}{B^i}\sum_{n\geq 0}\xi_n(a)\xi_n(b^i)\int_0^t \ell(\eta)e^{-\mu_n(t-\eta)}d\eta,\quad \forall t\in(0,T) \tag{23}$$

Therefore, given time records $d_i(t)$ in $(0,T)$ of the state u solution of the system (1)-(4) at a measuring well $b^i\in\Omega\setminus\{a\}$ and since (μ_n) is an increasing sequence that tends to $+\infty$, for minimising $\left\|u(b^i,t)-d_i(t)\right\|^2_{L^2(0,T)}$ with respect to ψ and ρ, one could truncate the series in (23) using a sufficiently large number N of first terms and solve:

$$\min_{\psi,\rho}\mathcal{R}_i^N(\psi,\rho):=\frac{1}{2}\left\|\frac{1}{B^i}\sum_{n=0}^{N}\xi_n(a)\xi_n(b^i)\int_0^t \ell(\eta)e^{-\mu_n(t-\eta)}d\eta-d_i(t)\right\|^2_{L^2(0,T)} \tag{24}$$

The solutions ψ and ρ obtained from minimising the sum over all measuring wells $b^{i=1,\dots,I}$ of the residuals $\mathcal{R}_i^N$ in (24) lead to determine the coefficients B^i in (18)-(19) that yield, according to Lemma 2.4, $S(a)S(b^i)=\left(B^i\right)^2$ for $i=1,\dots,I$. However, as far as the form of B^i used in (24) is concerned, the two main advantages of using (19) consist of: **1.** Avoiding the approximation of $U(\cdot,T)$ and $G_{b^i}(a)$ done by truncating the series defining B^i in (18) **2.** Since from (12) it holds $G_{b^i}(a)=G_a(b^i)$ for all $b^i\in\Omega\setminus\{a\}$, the coefficients $G_{b^i}(a)$ in (19) can be computed from solving numerically only one time the system (11) with a instead of b^i to determine $G_a(x_1,x_2)$, for all $(x_1,x_2)\in\Omega$.

Nevertheless, using (19) to compute the coefficients B^i requires the solution U of the system (15) to fulfill $U(\cdot,T)=0$ a.e. in Ω. To this end, let $\varphi_m(t)=\sqrt{2/T}\sin(m\pi t/T)$ for all $t\in(0,T)$ and $m\in\mathbb{N}^*$. It is well known that the set $\{\varphi_m\}$ forms a complete orthonormal family of $L^2(0,T)$. Then, we establish the following technical result:

Proposition 2.6 *Let $a\in\Omega$, $M\in\mathbb{N}^*$ and $\ell(t)=\sum_{m=1}^{M}\ell_m\varphi_m(t),\forall t\in(0,T)$. From using in the system (15) the pumping source $f(x_1,x_2,t)=\ell(t)\delta_a(x_1,x_2)$, it follows that its final state $U_N(\cdot,T)$ given by truncating the series in (17) to the order $N\in\mathbb{N}^*$ is subject to:*

$$\|U_N(\cdot,T)\|_{L^2(\Omega)}=\frac{1}{\sqrt{S(a)}}\|AX\|_2 \tag{25}$$

where $X = \big(\ell_1, \ldots, \ell_M\big)^\top \in I\!R^M$ *and* A *is the* $(N+1) \times M$ *matrix defined by*

$$A_{nm} = \xi_n(a)\frac{\sqrt{2T}}{m\pi}\,\frac{e^{-\mu_n T} - (-1)^m}{1 + \Big(\dfrac{T\mu_n}{m\pi}\Big)^2}, \qquad \textit{for } n = 0, \ldots, N;\ m = 1, \ldots, M \tag{26}$$

In (26), ξ_n *and* μ_n *are the normalized eigenfunctions and eigenvalues solutions of (10).*

Proof. For all $n \in \{0, \ldots, N\}$ and $m \in \{1, \ldots, M\}$, let $I_{nm} = \displaystyle\int_0^T \varphi_m(t)e^{-\mu_n(T-t)}dt$. Then, using twice an integration by parts, we get

$$I_{nm} = \frac{\sqrt{2T}}{m\pi}\,\frac{e^{-\mu_n T} - (-1)^m}{1 + \Big(\dfrac{T\mu_n}{m\pi}\Big)^2} \tag{27}$$

Besides, by employing in the system (15) the pumping source $f(x_1, x_2, t) = \ell(t)\delta_a(x_1, x_2)$ where $\ell(t) = \displaystyle\sum_{m=1}^{M} \ell_m\varphi_m(t), \forall t \in (0, T)$, it follows from truncating the series in (17) to the order $N \in I\!N^*$ that the final state $U_N(\cdot, T)$ of (15) satisfies:

$$\begin{aligned}
\|U_N(\cdot, T)\|^2_{L^2(\Omega)} &= \int_\Omega \Big(\sum_{n=0}^{N} \xi_n(x_1, x_2)\,\frac{\xi_n(a)}{\sqrt{S(a)}} \sum_{m=1}^{M} I_{nm}\ell_m\Big)^2 \\
&= \frac{1}{S(a)} \sum_{n=0}^{N}\sum_{k=0}^{N} \xi_n(a)\xi_k(a) \sum_{m=1}^{M} I_{nm}\ell_m \sum_{m=1}^{M} I_{km}\ell_m \int_\Omega \xi_n\xi_k \\
&= \frac{1}{S(a)} \sum_{n=0}^{N} \Big(\sum_{m=1}^{M} \xi_n(a)I_{nm}\ell_m\Big)^2 \\
&= \frac{1}{S(a)} \sum_{n=0}^{N} \Big(\sum_{m=1}^{M} A_{nm}\ell_m\Big)^2
\end{aligned} \tag{28}$$

where $A_{nm} = \xi_n(a)I_{nm}$ are the entries of the $(N+1) \times M$ matrix A involved in (25). Moreover, from replacing I_{nm} by its value obtained in (27), we find the entries A_{nm} announced in (26). Notice that the third equality in (28) is obtained since the set $\{\xi_n\}$ forms an orthonormal family of $L^2(\Omega)$. ■

Therefore, according to Proposition 2.6, in order to drive the solution U of the system (15) towards $U(\cdot, T) = 0$ a.e. in Ω, we employ the pumping source $f(x_1, x_2, t) = \ell(t)\delta_a(x_1, x_2)$ defined by $\ell(t) = \sum_{m=1}^{M} \ell_m\varphi_m(t), \forall t \in (0, T)$, where the coefficients ℓ_m are the components of the vector $X = \big(\ell_1, \ldots, \ell_M\big)^\top$ solution of the following minimisation problem:

$$\min_{X \in I\!R^M} \frac{1}{2}\big\|AX\big\|_2^2 \qquad \text{subject to:} \quad \sum_{m=1}^{M} \ell_m = 1 \tag{29}$$

In (29), $A = \big(A_{nm}\big)$ is the $(N+1) \times M$ matrix defined in (26).

3 Identifiability

We study the identifiability of the two unknown auxiliary variables ψ, ρ in (8) and of the unknown storativity function S using the observation operator $M[S, P]$ introduced in (6). We start by proving that $M[S, P]$ yields uniqueness of ρ when this latest is a real number, and of the unknown values of the storativity function at the measuring wells times its value at the pumping well. Then, under two additional assumptions, we establish a second result that yields identifiability for a wider class of unknown functions S and P.

Theorem 3.1 *Let $\ell \in L^2(0,T)$ be such that $\ell(t) \neq 0$ a.e. in $(0,T)$, $a \in \Omega$ be a pumping well and $b^{i=1,\dots,I} \in \Omega \setminus \{a\}$ be $I \in \mathbb{N}^*$ measuring wells. For all unknown functions S and P occurring in the problem (1)-(4) that are elements of the admissible set (3) and for which ρ in (8) is a real number, the observation operator $M[S, P]$ introduced in (6) determines uniquely the unknown values of ρ and of $S(a)S(b^i)$, for $i = 1, \dots, I$.*

Proof. Let $S^{(k=1,2)}$, $P^{(k=1,2)}$ be elements of the admissible set (3) and $u^{(k)}$ be the solution of the system (1)-(4) with $S^{(k)}$ and $P^{(k)}$ instead of S and P. Besides, let $\psi^{(k)}$ be the function and $\rho^{(k)}$ be the real number defined from $S^{(k)}$ and $P^{(k)}$ as in (8). We denote by $(\mu_n^{(k)})$ the eigenvalues and by $\{\xi_n^{(k)}\}$ the normalized eigenfunctions of the problem (10) with $\psi^{(k)}$ and $\rho^{(k)}$ instead of ψ and ρ. Since $(\mu_n^{(k)})$ is an increasing sequence of real numbers that tends to $+\infty$, it follows that the series in (17) defining $U^{(k)} = \sqrt{S^{(k)}}u^{(k)}$ converges uniformly in $]\tau, +\infty[$, for all $\tau > 0$. Therefore, $u^{(k)}$ can be written under the form

$$u^{(k)}(x_1, x_2, t) = \int_0^t \ell(\eta)\Phi^{(k)}(x_1, x_2, t-\eta)d\eta, \quad \forall (x_1, x_2, t) \in \Omega \times (0,T) \tag{30}$$

where the kernel $\Phi^{(k)}$ is defined in $\Omega \times (0,T)$ by

$$\Phi^{(k)}(x_1, x_2, t) = \sum_{n \geq 0} \bar{\xi}_n^{(k)}(a)\bar{\xi}_n^{(k)}(x_1, x_2)e^{-\mu_n^{(k)}t} \qquad \text{with: } \bar{\xi}_n^{(k)}(x_1, x_2) = \frac{\xi_n^{(k)}(x_1, x_2)}{\sqrt{S^{(k)}(x_1, x_2)}} \tag{31}$$

Let $M[S^{(k)}, P^{(k)}]$ be the observation operator defined as in (6) from recording in $(0,T)$ the state $u^{(k)}$ at the measuring wells $b^{i=1,\dots,I}$. Thus, we have

$$M[S^{(2)}, P^{(2)}] = M[S^{(1)}, P^{(1)}] \implies u^{(2)}(b^i, t) = u^{(1)}(b^i, t), \ \forall t \in (0,T), \text{ for } i = 1, \dots, I \tag{32}$$

Afterwards, according to (30)-(31), the assertion (32) yields: For $i = 1, \dots, I$

$$\int_0^t \ell(\eta)\Big(\Phi^{(2)}(b^i, t-\eta) - \Phi^{(1)}(b^i, t-\eta)\Big)d\eta = 0, \ \ \forall t \in (0,T) \tag{33}$$

Since the forcing function is such that $\ell(t) \neq 0$ a.e. in $(0,T)$ and using Titchmarsh's Theorem on convolution of L^1 functions [31], it follows from (33) that $\Phi^{(2)}(b^i, t) - \Phi^{(1)}(b^i, t) = 0$ a.e. in $(0,T)$. Hence, in view of (31), that leads to: For $i = 1, \dots, I$

$$\sum_{n \geq 0} \Big(\bar{\xi}_n^{(2)}(a)\bar{\xi}_n^{(2)}(b^i)e^{-\mu_n^{(2)}t} - \bar{\xi}_n^{(1)}(a)\bar{\xi}_n^{(1)}(b^i)e^{-\mu_n^{(1)}t}\Big) = 0, \quad \text{a.e. in } (0,T) \tag{34}$$

Moreover, since $\big(\mu_n^{(k=1,2)}\big)$ are both increasing sequences of real numbers that tend to infinity then, the series in (34) converges uniformly in $]\tau, +\infty[$, for all $\tau > 0$. Thus, this series defines a real-valued analytic function of $t \in]0,+\infty[$. Therefore, in view of (34) and by analytic extension, we get: For $i = 1,\dots,I$

$$
\begin{aligned}
& e^{-\mu_0^{(2)} t}\Big(\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^i) - \bar{\xi}_0^{(1)}(a)\bar{\xi}_0^{(1)}(b^i)e^{-(\mu_0^{(1)}-\mu_0^{(2)})t}\Big) \\
& + e^{-\mu_0^{(2)} t}\sum_{n\geq 1}\Big(\bar{\xi}_n^{(2)}(a)\bar{\xi}_n^{(2)}(b^i)e^{-(\mu_n^{(2)}-\mu_0^{(2)})t} - \bar{\xi}_n^{(1)}(a)\bar{\xi}_n^{(1)}(b^i)e^{-(\mu_n^{(1)}-\mu_0^{(2)})t}\Big) = 0,\ \forall t>0
\end{aligned}
\tag{35}
$$

Furthermore, in view of Remark 2.2, it follows that the principal eigenvalue of the problem (10) is unique i.e., $\mu_0^{(1)} < \mu_n^{(1)}$ and $\mu_0^{(2)} < \mu_n^{(2)}$, for all $n \in I\!N^*$.

Suppose that $\mu_0^{(1)} \neq \mu_0^{(2)}$. Say, for example, it holds $\mu_0^{(1)} > \mu_0^{(2)}$ which implies that we have also $\mu_n^{(1)} > \mu_0^{(2)}$, $\forall n \in I\!N^*$. Otherwise, we put in (35) rather $e^{-\mu_0^{(1)}t}$ in factor. Afterwards, in (35) from cancelling out $e^{-\mu_0^{(2)}t}$ and setting the limit when t tends to $+\infty$, we obtain $\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^i) = 0$. That is absurd since in Ω, $S^{(2)} > 0$ and, from Remark 2.2, we have $\xi_0^{(2)} = 1/\sqrt{\mathcal{S}(\Omega)}$. Hence, it follows that $\mu_0^{(1)} = \mu_0^{(2)}$. Then, by setting in (35) $\mu_0^{(1)} = \mu_0^{(2)}$, cancelling out $e^{-\mu_0^{(2)}t}$ and reevaluating the limit when t tends to $+\infty$, we find

$$
\begin{cases}
\mu_0^{(2)} = \mu_0^{(1)} \quad \Leftrightarrow \quad \rho^{(2)} = \rho^{(1)} \\
\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^i) = \bar{\xi}_0^{(1)}(a)\bar{\xi}_0^{(1)}(b^i) \quad \Leftrightarrow \quad S^{(2)}(a)S^{(2)}(b^i) = S^{(1)}(a)S^{(1)}(b^i), \ \text{for } i=1,\dots,I
\end{cases}
\tag{36}
$$

The two equivalence results in (36) are obtained from Remark 2.2. ■

Remark 3.2 *Provided the unknown functions S and P occurring in the problem (1)-(4) are elements of the admissible set (3) and such that their ρ in (8) is a real number, it follows from Theorem 3.1 that the observation operator $M[S,P]$ determines uniquely ρ and the coefficients B^i in (18) that yield $S(a)S(b^i) = \big(B^i\big)^2$, for $i = 1,\dots,I$. Besides, if ψ in (8) is also a real number then, provided μ_1 is of multiplicity 1 and $\xi_1(a)\xi_1(b^{i_0}) \neq 0$ where $i_0 \in \{1,\dots,I\}$, it follows by employing similar techniques as in (35)-(36) that $\mu_1^{(2)} = \mu_1^{(1)}$. Since $\rho^{(2)} = \rho^{(1)}$, that implies $\psi^{(2)} = \psi^{(1)}$ and thus, $M[S,P]$ yields also uniqueness of ψ.*

As far as the identifiability of the unknown auxiliary variables ψ and ρ in (8) for a wider class of unknown functions S and P is concerned, we establish the following result:

Theorem 3.3 *In the problem (1)-(4), provided $\ell \in L^2(0,T)$ satisfying $\ell(t) \neq 0$ a.e. in $(0,T)$ and the unknown functions S and P are elements of (3) generating auxiliary variables ψ and ρ in (8) such that the eigenpairs solutions of the system (10) fulfill:*

1. *The eigenvalues μ_n are distinct, for all $n \in I\!N$. That implies (μ_n) is a strictly increasing sequence of real numbers that tends to $+\infty$.*
2. *There exists a pumping position $a \in \Omega$ and a measuring position $b^{i_0} \in \Omega \setminus \{a\}$ that are both strategic with respect to $\{\xi_n\}$.*

The observation operator $M[S,P]$ introduced in (6) yields uniqueness for all $n \in I\!N$ of μ_n and of $\bar{\xi}_n(a)\bar{\xi}_n(b^i)$ for $i = 1,\dots,I$, where $\bar{\xi}_n = \xi_n/\sqrt{S}$ in Ω.

Proof. We use the same notations employed in the proof of Theorem 3.1 and assume that the two sequences $(\mu_n^{(k=1,2)})$ are both strictly increasing whereas the pumping position a and the measuring position b^{i_0} are both strategic with respect to $\{\xi_n^{(k=1,2)}\}$.

- *For $i = i_0$ in (35):* Suppose that $\mu_0^{(1)} \neq \mu_0^{(2)}$. Say, for example, $\mu_0^{(1)} > \mu_0^{(2)}$ which since both sequences $(\mu_n^{k=1,2})$ are strictly increasing implies that $\mu_n^{(1)} > \mu_0^{(2)}, \forall n \geq 1$. If $\mu_0^{(1)} < \mu_0^{(2)}$ then, we put in (35) rather $e^{-\mu_0^{(1)}t}$ in factor. From cancelling out the term $e^{-\mu_0^{(2)}t}$ then, setting the limit when t tends to $+\infty$ in (35), we get $\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^{i_0}) = 0$. That is absurd since a and b^{i_0} are both strategic with respect to $\{\xi_n^{(k=1,2)}\}$. Hence, $\mu_0^{(1)} = \mu_0^{(2)}$. In (35), by setting $\mu_0^{(1)} = \mu_0^{(2)}$ and cancelling out the term $e^{-\mu_0^{(2)}t}$ then, reevaluating the limit when t tends to $+\infty$, we find $\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^{i_0}) = \bar{\xi}_0^{(1)}(a)\bar{\xi}_0^{(1)}(b^{i_0})$. Therefore, for $i = i_0$ the term associated with $n = 0$ in the series (34) vanishes. By iterating the same process for all $n \geq 1$, we obtain

$$\forall n \in I\!N, \qquad \mu_n^{(1)} = \mu_n^{(2)} \quad \text{and} \quad \bar{\xi}_n^{(1)}(a)\bar{\xi}_n^{(1)}(b^{i_0}) = \bar{\xi}_n^{(2)}(a)\bar{\xi}_n^{(2)}(b^{i_0}) \tag{37}$$

- *Setting $\mu_n^{(1)} = \mu_n^{(2)} = \mu_n, \forall n \in I\!N$ in (34):* It follows by analytic extension and putting in factor $e^{-\mu_0 t}$ that: For $i = 1, \ldots, I$ and all $t > 0$,

$$e^{-\mu_0 t}\left(\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^i) - \bar{\xi}_0^{(1)}(a)\bar{\xi}_0^{(1)}(b^i) + \sum_{n\geq 1}\left(\bar{\xi}_n^{(2)}(a)\bar{\xi}_n^{(2)}(b^i) - \bar{\xi}_n^{(1)}(a)\bar{\xi}_n^{(1)}(b^i)\right)e^{-(\mu_n-\mu_0)t}\right) = 0 \tag{38}$$

Afterwards, in (38), from cancelling out the term $e^{-\mu_0 t}$ then, setting the limit when t tends to $+\infty$, we find $\bar{\xi}_0^{(2)}(a)\bar{\xi}_0^{(2)}(b^i) = \bar{\xi}_0^{(1)}(a)\bar{\xi}_0^{(1)}(b^i)$. Moreover, iterating the same process for all $n \geq 1$ leads to

$$\forall n \in I\!N, \qquad \bar{\xi}_n^{(1)}(a)\bar{\xi}_n^{(1)}(b^i) = \bar{\xi}_n^{(2)}(a)\bar{\xi}_n^{(2)}(b^i), \quad \text{for } i = 1, \ldots, I \tag{39}$$

Therefore, from (37) and (39), it follows that

$$\forall n \in I\!N, \qquad \mu_n^{(1)} = \mu_n^{(2)} \quad \text{and} \quad \bar{\xi}_n^{(1)}(a)\bar{\xi}_n^{(1)}(b^i) = \bar{\xi}_n^{(2)}(a)\bar{\xi}_n^{(2)}(b^i), \ \text{for } i = 1, \ldots, I \tag{40}$$

The results in (40) are those announced in Theorem 3.3. ■

Remark 3.4 *The analysis of the results in Theorem 3.3 leads to point out the two following remarks that yield uniqueness of $S(a)S(b^i)$, for $i = 1, \ldots, I$ as well as of ψ and ρ for a wide class of unknown functions S and P:*

1. *Since $\bar{\xi}_n^{(k)} = \xi_n^{(k)}/\sqrt{S^{(k)}}$ in Ω and the second equality in (40) holds for all $n \in I\!N$, we believe that this equality would imply that $S^{(2)}(a)S^{(2)}(b^i) = S^{(1)}(a)S^{(1)}(b^i)$, for $i = 1, \ldots, I$ in a large class of auxiliary variables ψ and ρ. Notice that according to Theorem 3.1, the implication holds true when ρ is a real number, for all ψ.*

2. *Provided $S^{(2)}(a)S^{(2)}(b^i) = S^{(1)}(a)S^{(1)}(b^i)$, for $i = 1, \ldots, I$ and using a sufficiently large number I of measuring wells, the second equality in (40) implies that there exists a subset $\mathcal{N} \subset I\!N$ such that: $\xi_n^{(1)} = \xi_n^{(2)}$ a.e. in Ω, for all $n \in \mathcal{N}$. Afterwards, by setting $\xi_n^{(1)} = \xi_n^{(2)} = \xi_n$ and since $\mu_n^{(1)} = \mu_n^{(2)}$, it follows from (10) that*

$$\int_\Omega \left(\psi^{(2)} - \psi^{(1)}\right)\left\|\nabla \xi_n\right\|_2^2 + \int_\Omega \left(\rho^{(2)} - \rho^{(1)}\right)\xi_n^2 = 0, \quad \forall n \in \mathcal{N} \tag{41}$$

For example, when $\rho^{(k=1,2)}$ are two real numbers, it follows from Theorem 3.1 that $\rho^{(2)} = \rho^{(1)}$. In this case, provided $\mathcal{N}$ is a non-empty set and $\left\|\nabla \xi_n\right\|_2^2 \neq 0$ a.e. in Ω, the result (41) yields uniqueness of ψ in the class of functions whose the difference is a function that keeps a constant sign a.e. in Ω.

4 Identification method

In this section, using the observation operator $M[S, P]$ introduced in (6), we develop an identification method that leads to determine the two unknown functions S and P occurring in the problem (1)-(4). Under some conditions on the number and the locations of the employed measuring wells, the developed method starts by determining the two unknown intermediate variables ψ and ρ in (8) from minimising the sum over all measuring wells of the residuals $\mathcal{R}_i^N$ defined by (24). Then, the determined ψ and ρ lead to compute the coefficients B^i in (18)-(19) that, according to Lemma 2.4, yield $S(a)S(b^i) = \big(B^i\big)^2$ for $i = 1, \ldots, I$. Afterwards, we reconstruct the unknown function S and thus, in view of (8), deduce $P = S\psi$. To this end, we propose the two following ways for the reconstruction of the unknown storativity function S in Ω:

4.1 Local determination of S

This first way of determining the unknown function S is based on assuming that it holds $\operatorname{div}(\psi\Psi) = 0$ in Ω. In fact, from dividing the equations defining the problem (1)-(4) by S, it follows that the state u solves also the system:

$$\begin{cases} \partial_t u - \operatorname{div}\big(\psi\nabla u\big) - \psi\Psi\nabla u = \dfrac{\ell(t)}{S(a)}\delta_a(x_1, x_2) & \text{in } \Omega \times (0, T) \\ u(\cdot, 0) = 0 \qquad \text{in } \Omega \\ \psi\nabla u \cdot \nu = 0 \qquad \text{on } \partial\Omega \times (0, T) \end{cases} \tag{42}$$

where the vector field $\psi\Psi$ stands for the advection term. Therefore, the sense of the assumption $\operatorname{div}(\psi\Psi) = 0$ in Ω follows from the fact that water is an incompressible fluid. Moreover, in view of (8), this assumption reduces the equation satisfied by the unknown function S into the following first order nonlinear partial differential equation:

$$\Big(\frac{\partial_{x_1} S}{S}\Big)^2 + \Big(\frac{\partial_{x_2} S}{S}\Big)^2 = 4\frac{\rho}{\psi} \quad \text{in } \Omega \tag{43}$$

Since the two variables ψ and ρ have been already identified, it follows that the equation (43) could inform us about the local distribution in Ω of the unknown function S. Thus, once the coefficients B^i in (18)-(19) are computed from the identified ψ and ρ, we establish a local determination procedure of the function S that combines the knowledge on its distribution obtained from (43) with the knowledge in (18) of $S(a)S(b^i) = (B^i)^2$, for $i = 1, \ldots, I$. This local determination proceeds as follows:

Algorithm: Local determination

1. Find an open subset $\omega_0 \subseteq \Omega$ that contains a measuring well b^{i_0}, the pumping well a and within which $\rho = 0$. It follows from (43) that $\nabla S = \vec{0}$ in ω_0. Since $S(a)S(b^{i_0}) = (B^{i_0})^2$ then, $S(x_1, x_2) = B^{i_0}$ in ω_0 and thus, $S(a) = B^{i_0}$. Furthermore, from dividing the known products $S(a)S(b^i)$ by $S(a)$, we deduce the value of $S(b^i)$, for $i = 1, \ldots, I$.

2. For all open subset $\omega \subset \Omega$ containing a measuring well $b^i = (b_1^i, b_2^i)$ and within which the identified function ρ/ψ admits symmetric variations: There exists two real numbers

α, β and a non-negative derivable real-valued function $h : I\!R \longrightarrow I\!R_+$ such that

$$\frac{\rho\big(x_1,x_2\big)}{\psi\big(x_1,x_2\big)} = h\big(\alpha(x_1+x_2)+\beta\big) \quad \text{in } \omega \tag{44}$$

which implies that it holds

$$\partial_{x_1}\left(\frac{\rho(x_1,x_2)}{\psi(x_1,x_2)}\right) = \partial_{x_2}\left(\frac{\rho(x_1,x_2)}{\psi(x_1,x_2)}\right) \quad \text{in } \omega \tag{45}$$

the solution in ω of the equation (43) is given by

$$S(x_1,x_2) = S(b^i)\exp\left(\int_{b_1^i}^{x_1}\sqrt{2\frac{\rho(\eta,x_2)}{\psi(\eta,x_2)}}d\eta + \int_{b_2^i}^{x_2}\sqrt{2\frac{\rho(b_1^i,\zeta)}{\psi(b_1^i,\zeta)}}d\zeta\right) \quad \text{in } \omega \tag{46}$$

where exp stands for the exponential function. According to (44)-(45), it follows that the solution S obtained in (46) applies, in particular, for all region $\omega \subset \Omega$ where it holds either $\rho = 0$ or ρ/ψ is equal to a constant. Moreover, in these two particular cases, from (46) it comes that: For all $\omega \subset \Omega$ containing a measuring well $b^i = (b_1^i, b_2^i)$ within which

$$\begin{aligned} &\bullet\ \rho = 0 \quad \text{in } \omega && \Longrightarrow \quad S(x_1,x_2) = S(b^i) \quad \text{in } \omega \\ &\bullet\ \nabla\Big(\frac{\rho}{\psi}\Big) = \vec{0} \quad \text{in } \omega && \Longrightarrow \quad S(x_1,x_2) = S(b^i)\exp\left(\sqrt{2\frac{\rho}{\psi}}\Big(x_1 - b_1^i + x_2 - b_2^i\Big)\right) \quad \text{in } \omega \end{aligned} \tag{47}$$

However, if (45) doesn't apply then, in view of (18) and (43), we search for S from solving:

$$\begin{cases} \displaystyle\min_{S>0}\frac{1}{2}\left\|\Big(\frac{\partial_{x_1}S}{S}\Big)^2 + \Big(\frac{\partial_{x_2}S}{S}\Big)^2 - 4\frac{\rho}{\psi}\right\|^2_{L^2(\omega)} \\ \text{Subject to:} \quad S(a)S(b^i) = \big(B^i\big)^2, \text{ for all } b^i \in \omega \end{cases} \tag{48}$$

Remark 4.1 *The existence of the subset ω_0 in* **1.** *could be ensured by setting a measuring well b^{i_0} as close as possible to the pumping well a in order to get these two wells lying in a small region of Ω where the storativity S remains constant.*

4.2 Global determination of S

By searching for a solution to the equation (9) under the form $S(x_1,x_2) = e^{G(x_1,x_2)}$ in Ω, it follows that the unknown function G should solve the second order elliptic nonlinear partial differential equation with gradient terms defined by:

$$\Delta G + \frac{1}{2}\|\nabla G\|_2^2 + \frac{1}{\psi}\nabla\psi\cdot\nabla G = 2\frac{\rho}{\psi} \quad \text{in } \Omega \tag{49}$$

In the litterature, solving (49) appears to be a challenging task since the existence and the behaviour near the boundary $\partial\Omega$ of its solutions rely on the growth of $(1/\psi)\nabla\psi$ in Ω, the regularity of $\partial\Omega$ and of $2\rho/\psi$, see [2, 23]. Therefore, in view of (18)-(19), we determine rather an approximation g of the unknown function G based on the knowledge of $e^{G(a)+G(b^i)} = (B^i)^2$, for $i = 1, \ldots, I$. Thus, we consider that $S(x_1, x_2) \approx e^{g(x_1,x_2)}$ in Ω, where the unknown function g is a real polynomial subject to:

$$g(a) + g(b^i) = 2\ln(B^i), \quad \text{for } i = 1, \ldots, I \tag{50}$$

Hence, (50) yields a system of I linear equations on the $N_g \in \mathbb{N}^*$ unknown coefficients defining the sought polynomial g. Provided $I \geq N_g$ and the measuring positions $b^{i=1,\ldots,I}$ are set in $\Omega \setminus \{a\}$ such that N_g equations of (50) are linearly independent, the unknown polynomial g is uniquely determined from (50). For example, in the case when $I \geq N_g = 3$ i.e., g is a real polynomial of degree 1: $g(x_1, x_2) = g_1 x_1 + g_2 x_2 + g_0$, it follows from (50) that the unknown coefficients g_0, g_1 and g_2 defining g are subject to:

$$\begin{pmatrix} a_1 + b_1^1 & a_2 + b_2^1 & 2 \\ a_1 + b_1^2 & a_2 + b_2^2 & 2 \\ a_1 + b_1^3 & a_2 + b_2^3 & 2 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_0 \end{pmatrix} = 2 \begin{pmatrix} \ln(B^1) \\ \ln(B^2) \\ \ln(B^3) \end{pmatrix} \tag{51}$$

The determinant of the 3×3 matrix in (51) is: $b_2^3\big(b_1^2 - b_1^1\big) + b_1^3\big(b_2^1 - b_2^2\big) + b_1^1 b_2^2 - b_2^1 b_1^2$. Selecting the positions of the measuring wells b^1, b^2 and b^3 in a way that affects this determinant to be non-null leads to uniquely determine the unknown coefficients g_0, g_1 and g_2 from solving the linear system (51). Thus, the global determination proceeds as follows:

Algorithm: Global determination

Begin

1. Let g be a *desired* polynomial defined by $N_g \in \mathbb{N}^*$ unknown coefficients.
Set the measuring wells in the domain $\Omega \setminus \{a\}$ such that:

i) Their number $I \geq N_g$.

ii) Their positions yield: N_g equations of (50) are linearly independent.

2. Use the identified ψ and ρ to compute B^i in (18)-(19), for $i = 1, \ldots, I$.

3. Determine g from solving N_g linearly independent equations of (50).

4. Set $S(x_1, x_2) \approx e^{g(x_1,x_2)}$ in Ω.

End

Therefore, provided the number I and the positions of the measuring wells $b^{i=1,\ldots,I}$ fulfill **i)** and **ii)**, the global determination gives an approximation of the unknown storativity S i.e., $S(x_1, x_2) \approx e^{g(x_1,x_2)}$, where g is a real polynomial of degree up to the user.

4.3 Procedure for the identification of S and P

For the clarty of our presentation, in this subsection we summarize the main steps defining the developed identification method. Let $\{\zeta_j\}$ be a complete orthonormal family of $L^2(\Omega)$ and $(J, M, N) \in (\mathbb{N}^*)^3$ be sufficiently large numbers of first terms. The identification of the two unknown functions S and P proceeds in the following four steps:

- **Step 1.** Select the pumping source and the measuring wells by fulfilling:

1. Hydraulic forcing function $\ell \in L^2(0,T)$ such that Theorem 3.1 applies.

2. Pumping position $a \in \Omega$ and a measuring position $b^{i_0} \in \Omega \setminus \{a\}$ such that Remark 4.1 applies. This is required for using Local Determination of S.

3. The number I of measuring wells b^i and their positions in $\Omega \setminus \{a\}$ such that Algorithm Global Determination applies.

Employ the pumping source $f(x_1,x_2,t) = \ell(t)\delta_a(x_1,x_2)$ in $\Omega \times (0,T)$. Record in $(0,T)$ the resulting drawdown $d_i(t) = u(b^i,t)$ at the measuring well b^i, for $i = 1,\dots,I$.

- **Step 2.** In the eigenvalue problem (10), set $\psi(x_1,x_2) = \sum_{j=0}^{J} \psi_j \zeta_j(x_1,x_2)$. Determine the coefficients $\psi_0,\dots,\psi_J$ and ρ that solve the following minimisation problem:

$$\min_{\psi_0,\dots,\psi_J,\rho} \sum_{i=1}^{I} \mathcal{R}_i^N(\psi,\rho) \qquad \text{subject to: } \sum_{j=0}^{J} \psi_j \zeta_j(x_1,x_2) > 0 \text{ in } \Omega \tag{52}$$

where $\mathcal{R}_i^N$ is the residual, associated to the measuring well b^i, introduced in (24).

- **Step 3.** Identification of the two unknown functions S and P:
  - ▶ Computation of the coefficients B^i that yield $S(a)S(b^i) = \left(B^i\right)^2$, for $i = 1,\dots,I$:
    - ⋆ **Option 1.** Use ψ and ρ determined in step 2 to compute the coefficients B^i for $i = 1,\dots,I$ from (18).
    - ⋆ **Option 2.** Using ψ and ρ identified in step 2 do:
      1. Apply Proposition 2.6 to select $\ell \in L^2(0,T)$ such that $\ell(t) \neq 0$ a.e. in (0,T) and yields $U(\cdot,T) = 0$ a.e. in Ω.
      2. Reforce the system with ℓ and record $d_i(t) = u(b^i,t), \forall t \in (0,T)$ for $i = 1,\dots,I$.
      3. Compute the coefficients $B^{i=1,\dots,I}$ from (19).
  - ▶ Determine the unknown function S in Ω using the local or the global determination.

- **Step 4.** Deduce the unknown function $P(x_1,x_2) = S(x_1,x_2)\sum_{j=0}^{J} \psi_j \zeta_j(x_1,x_2)$ in Ω.

Remark 4.2 *Regarding the developed identification method, we point out the following:*

1. *In step 3., computing the coefficients $B^{i=1,\dots,I}$ from (19) enables to avoid the approximation done by truncating the series in (18). Moreover, according to (12), the coefficients $G_{b^i}(a)$ in (19) can be computed for $i = 1,\dots,I$ from solving only once the problem (11) with a instead of b^i to compute $G_a(x_1,x_2), \forall (x_1,x_2) \in \Omega$. Then, the symmetric property (12) implies that $G_{b^i}(a) = G_a(b^i)$, for $i = 1,\dots,I$.*

2. *To apply the developed identification method for determining ρ as an unknown function, one could consider in the eigenvalue problem (10) that $\rho(x_1,x_2) = \sum_{j=0}^{J} \rho_j \zeta_j(x_1,x_2)$ and solves (52) with respect to the unknown coefficients $\psi_0,\dots,\psi_J$ and $\rho_0,\dots,\rho_J$.*

5 Numerical experiments

We apply the developed identification method to the case of a rectangular aquifer represented by the domain $\Omega := (0, L_1) \times (0, L_2)$, where $0 < L_2 \leq L_1$. This aquifer is characterized by unknown hydraulic storativity S and transmissivity P functions whose the intermediate variables ψ and ρ defined in (8) are two real unknown numbers.

For numerical purposes, due to the different ranges of the two unknown functions S and P which would lead to a significant range difference between the two optimisation variables ψ and ρ in (52), we derive the non-dimensional version of the results established in this paper. To this end, for all $(x_1, x_2, t) \in \Omega \times (0, T)$, let:

$$x = \frac{x_1}{L_1}, \quad y = \frac{x_2}{L_2} \quad \text{and} \quad s = \frac{t}{T} \tag{53}$$

That reduces the domain of study from $\Omega \times (0, T)$ into the unit cube $\big(0, 1\big)^3$. Moreover, let $\bar{u}(x, y, s) = u(xL_1, yL_2, sT) = u(x_1, x_2, t)$, $\bar{S}(x, y) = S(xL_1, yL_2) = S(x_1, x_2)$ and $\bar{P}(x, y) = P(xL_1, yL_2) = P(x_1, x_2)$, for all $(x, y, s) \in (0, 1)^3$. Then, u solves the problem (1)-(4) is equivalent to $\bar{u}$ satisfies the following system:

$$\begin{cases} \bar{S}\partial_s \bar{u} - \operatorname{div}\big(D\bar{P}\nabla\bar{u}\big) = T\bar{\ell}(s)\delta_{\bar{a}}(x, y) & \text{in } \big(0, 1\big)^3 \\ \bar{u}(x, y, 0) = 0 \quad \text{in } \big(0, 1\big)^2 \\ \bar{P}\nabla\bar{u} \cdot \nu = 0 \quad \text{on } \partial\Big(\big(0, 1\big)^2\Big) \times \big(0, 1\big) \end{cases} \tag{54}$$

where $\bar{\ell}(s) = \ell(sT) = \ell(t)$, $\delta_{\bar{a}}$ is the dirac mass at $\bar{a} = (\frac{a_1}{L_1}, \frac{a_2}{L_2}) \in \big(0, 1\big)^2$ and D is the diagonal 2×2 matrix defined by

$$D = \begin{pmatrix} \dfrac{T}{L_1^2} & 0 \\ 0 & \dfrac{T}{L_2^2} \end{pmatrix} \tag{55}$$

Afterwards, it follows that $\bar{U}(x, y, s) = \sqrt{\bar{S}(x, y)}\bar{u}(x, y, s)$ solves:

$$\begin{cases} \partial_s \bar{U} - \operatorname{div}\big(D\bar{\psi}\nabla\bar{U}\big) + \bar{\rho}\bar{U} = \dfrac{T\bar{\ell}(s)}{\sqrt{\bar{S}(\bar{a})}}\delta_{\bar{a}}(x, y) & \text{in } \big(0, 1\big)^3 \\ \bar{U}(x, y, 0) = 0 \quad \text{in } \big(0, 1\big)^2 \\ \bar{\psi}\nabla\bar{U} \cdot \nu = 0 \quad \text{on } \partial\Big(\big(0, 1\big)^2\Big) \times \big(0, 1\big) \end{cases} \tag{56}$$

where $\bar{\psi} = \bar{P}/\bar{S}$, $\bar{\Psi} = (1/\bar{S})\nabla\bar{S}$ and

$$\begin{aligned} \bar{\rho} &= \tfrac{1}{2}\operatorname{div}\big(D\bar{\psi}\bar{\Psi}\big) + \tfrac{1}{4}\bar{\psi}\bar{\Psi}D\bar{\Psi} \\ &= T\rho \end{aligned} \tag{57}$$

Remark 5.1 *Since $\bar{\psi} = \psi$, it follows from (57) that the non-dimensional version $\bar{\rho}$ of the auxiliary variable ρ in (8) could indicate how do the final monitoring time T should be selected to keep the two optimisation variables $\bar{\psi}$ and $\bar{\rho}$ having about the same range. That leads to enhance the minimisation of the non-dimensional version of $\mathcal{R}_i^N$ in (52).*

Then, we introduce the following associated eigenvalue problem:

$$\begin{cases} -\mathrm{div}\big(D\bar{\psi}\nabla\xi_n\big) + \bar{\rho}\xi_n = \mu_n\xi_n & \text{in } \big(0,1\big)^2 \\ \bar{\psi}\nabla\xi_n \cdot \nu = 0 \quad \text{on } \partial\Big(\big(0,1\big)^2\Big) \end{cases} \tag{58}$$

Since in the case of our numerical experiments $\bar{\psi}$ and $\bar{\rho}$ are two unknown real numbers, it follows that the eigenfunctions $\{\xi_{nm}\}$ and eigenvalues (μ_{nm}) solutions of (58) are

$$\xi_{nm}(x,y) = c_{nm}\cos\big(n\pi x\big)\cos\big(m\pi y\big) \quad \text{and} \quad \mu_{nm} = T\bar{\psi}\Big(\big(\frac{n\pi}{L_1}\big)^2 + \big(\frac{m\pi}{L_2}\big)^2\Big) + \bar{\rho} \tag{59}$$

for all $(x,y) \in \big(0,1\big)^2$, where $(n,m) \in I\!N^2$ and c_{nm} are normalizing coefficients:

$$c_{nm} = \begin{cases} 1 & \text{if } n = m = 0 \\ \sqrt{2} & \text{if } nm = 0 \text{ and } n+m > 0 \\ 2 & \text{if } nm \neq 0 \end{cases} \tag{60}$$

Remark 5.2 *From (59), $\mu_{n_1m_1} = \mu_{n_2m_2}$ implies that $L_2^2/L_1^2 = \big((m_2^2-m_1^2)/(n_1^2-n_2^2)\big) \in \mathbb{Q}$. Hence, by contraposition it follows that selecting L_1 and L_2 such that $L_2^2/L_1^2 \in I\!R \setminus \mathbb{Q}$ implies that all eigenvalues μ_{nm} in (59) are of multiplicity equal to* 1*.*

Besides, to generate synthetic measurements, we use: For all $(x,y) \in \big(0,1\big)^2$,

$$\bar{S}(x,y) = e^{-\gamma_1 L_1 x - \gamma_2 L_2 y - \gamma_0} \qquad \text{and} \qquad \bar{P}(x,y) = \bar{\psi}\bar{S}(x,y) \tag{61}$$

In (61), the coefficients γ_0, γ_1, γ_2 and $0 < \bar{\psi}$ are four real unknown numbers. Therefore, $\bar{\Psi} = -\big(\gamma_1 L_1, \gamma_2 L_2\big)^\top$ which, according to (57), implies that

$$\bar{\rho} = \frac{T\bar{\psi}}{4}\Big(\gamma_1^2 + \gamma_2^2\Big) \quad \Longrightarrow \quad \bar{\rho} = T\rho \tag{62}$$

The implication in (62) is obtained since $\bar{\psi} = \psi$ and $\Psi = -\big(\gamma_1, \gamma_2\big)^\top$. Given the pumping position $\bar{a} = (\bar{a}_1, \bar{a}_2) \in \big(0,1\big)^2$, we use the following definition of the dirac mass:

$$\delta_{\bar{a}}(x,y) = \frac{L_1L_2}{\pi}\lim_{\eta\longrightarrow 0^+}\frac{1}{\eta}e^{-\frac{L_1^2(x-\bar{a}_1)^2+L_2^2(y-\bar{a}_2)^2}{\eta}} \tag{63}$$

Afterwards, we determine $\bar{U}_F(x,y,s) = \dfrac{L_1L_2}{4\pi\bar{\psi}Ts}\mathcal{H}(s)e^{-\frac{L_1^2(x-\bar{a}_1)^2+L_2^2(y-\bar{a}_2)^2}{4\bar{\psi}Ts}-\bar{\rho}s}$ that solves

$$\partial_s\bar{U}_F - \bar{\psi}\mathrm{div}\big(D\nabla\bar{U}_F\big) + \bar{\rho}\bar{U}_F = \delta_0(s)\delta_{\bar{a}}(x,y) \quad \text{in } I\!R^2 \times I\!R \tag{64}$$

where $\delta_0(s)$ is the dirac mass at $s = 0$ and $\mathcal{H}$ is the Heaviside function [27]. Hence, the solution $\bar{U}$ of the system (56), where $\bar{\psi} > 0$ and $\bar{\rho}$ are two real numbers, is given by

$$\begin{aligned} \bar{U}(x,y,s) &= \frac{\mathcal{H}(s)T\bar{\ell}(s)}{\sqrt{\bar{S}(\bar{a})}} \star_s \bar{U}_F(x,y,s) + \bar{U}_0(x,y,s), \quad \forall (x,y,s) \in \big(0,1\big)^3 \\ &= \frac{L_1L_2}{4\pi\bar{\psi}\sqrt{\bar{S}(\bar{a})}}\int_0^s \bar{\ell}(s-\eta)\frac{1}{\eta}e^{-\frac{L_1^2(x-\bar{a}_1)^2+L_2^2(y-\bar{a}_2)^2}{4\bar{\psi}T\eta}-\bar{\rho}\eta}d\eta + \bar{U}_0(x,y,s) \end{aligned} \tag{65}$$

where $\star_s$ represents the convolution product with respect to the variable s and $\bar{U}_0$ solves

$$\begin{cases} \partial_s \bar{U}_0 - \bar{\psi}\mathrm{div}\big(D\nabla \bar{U}_0\big) + \bar{\rho}\bar{U}_0 = 0 \quad \text{in } \big(0,1\big)^3 \\ \bar{U}_0(x,y,0) = 0 \quad \text{in } \big(0,1\big)^2 \\ \nabla \bar{U}_0 \cdot \nu = -\frac{\mathcal{H}(s)T\bar{\ell}(s)}{\sqrt{\bar{S}(\bar{a})}} \star_s \nabla \bar{U}_F \cdot \nu \quad \text{on } \partial\Big(\big(0,1\big)^2\Big) \end{cases} \tag{66}$$

We employ the source forcing function $\bar{\ell}(s) = \ell_0 \sin\big(k\pi s\big), \forall s \in \big(0,1\big)$, where the coefficients $\ell_0 \in I\!R^*$ and $k \in I\!N^*$. Furthermore, to compute the non-dimensional version of the residuals $\mathcal{R}_i^N$ in (24) and their partial derivatives with respect to the optimisation variables $\bar{\psi}$ and $\bar{\rho}$, we verify that: For all $s \in \big(0,1\big]$,

$$\int_0^s \ell(\eta) e^{-\mu_{nm}(s-\eta)} d\eta = \frac{\ell_0}{\mu_{nm}^2 + \big(k\pi\big)^2}\Big(\mu_{nm}\sin\big(k\pi s\big) + k\pi\Big[e^{-\mu_{nm}s} - \cos\big(k\pi s\big)\Big]\Big) \tag{67}$$

Then, for all $X \in \{\bar{\psi}, \bar{\rho}\}$, it follows from (67) that

$$\partial_X\left(\int_0^s \ell(\eta)e^{-\mu_{nm}(s-\eta)}d\eta\right) = \frac{\ell_0\, \partial_X \mu_{nm}}{\Big(\mu_{nm}^2 + \big(k\pi\big)^2\Big)^2} Q_n^k(s), \quad \forall s \in \big(0,1\big] \tag{68}$$

where

$$Q_n^k(s) = \Big(\big(k\pi\big)^2 - \mu_{nm}^2\Big)\sin\big(k\pi s\big) + k\pi\Big(2\mu_{nm}\cos\big(k\pi s\big) - \Big(2\mu_{nm} + \Big[\mu_{nm}^2 + \big(k\pi\big)^2\Big]s\Big)e^{-\mu_{nm}s}\Big) \tag{69}$$

Moreover, we have

$$\int_0^1 \ell(\eta)\big(1 - e^{-\mu_{nm}(1-\eta)}\big)d\eta = \ell_0\left(\frac{1}{k\pi}\Big(1-(-1)^k\Big) - \frac{k\pi}{\mu_{nm}^2 + \big(k\pi\big)^2}\Big(e^{-\mu_{nm}} - (-1)^k\Big)\right) \tag{70}$$

Hence, in view of (23) and provided $U(x,T)$ doesn't vanish a.e. in Ω, we get

$$\begin{aligned} &\partial_X\left(\frac{1}{\mu_{nm}}\int_0^1 \ell(\eta)\big(1 - e^{-\mu_{nm}(1-\eta)}\big)d\eta\right) \\ &= -\frac{\partial_X \mu_{nm}}{\mu_{nm}^2}\left(\int_0^1 \ell(\eta)\big(1 - e^{-\mu_{nm}(1-\eta)}\big)d\eta + \frac{\ell_0 \mu_{nm}}{\Big(\mu_{nm}^2 + \big(k\pi\big)^2\Big)^2} Q_n^k(1)\right) \end{aligned} \tag{71}$$

We solved numerically the problem (66) using the five-point finite difference Crank-Nicolson scheme and generated state time records $\bar{d}_i(s)$, for all $s \in \big(0,1\big)$ and $i = 1, \dots, I$ from (65) such that $\bar{d}_i(s) = \bar{U}(\bar{b}^i, s)/\sqrt{\bar{S}(\bar{b}^i)}$, where $\bar{b}^i = (\frac{b_1^i}{L_1}, \frac{b_2^i}{L_2})$. Then, we solved the non-dimensional version of the minimisation problem (52) using the BFGS quasi-Newton method combined with Wolfe line search. In the sequel, we present the numerical results obtained from solving the non-dimensional version of (52).

- **Part1: Identification of the auxiliary variables $\bar{\psi}$ and $\bar{\rho}$**

We carried out numerical experiments using a pumping well located at $\bar{a} = (0.5, 0.5)$ in the non-dimensional domain $\big(0,1\big)^2$ and forcing the domain with $\bar{\ell}(s) = \ell_0 \sin(k\pi s)$ for all $s \in \big(0,1\big)$, where $\ell_0 = 1$ and $k = 4$. We used the first $N = M = 5$ eigenpairs of (58)-(60) and a total number of $N_t = 60$ measurements taken regularly i.e., with the uniform time step $\Delta t = T/N_t$ during the monitoring time T, at each of the measuring wells:

Measuring wells	$\bar{b}^1$	$\bar{b}^2$	$\bar{b}^3$	$\bar{b}^4$	$\bar{b}^5$	$\bar{b}^6$
Position in $\big(0,1\big)^2$	$(0.4, 0.6)$	$(0.6, 0.4)$	$(0.5, 0.2)$	$(0.6, 0.6)$	$(0.4, 0.4)$	$(0.5, 0.8)$

Table 1: Positions of the measuring wells in the non-dimensional domain $\big(0,1\big)^2$.

We initialized the two optimisation variables to $\bar{\psi}_j = 1$ and $\bar{\rho}_j = 10^{-6}$. Then, we solved the non-dimensional version of the minimisation problem (52) using measurements, taken at the I first measuring wells of Table 1, generated from the storativity function $\bar{S}$ introduced in (61) and different values of $\bar{\psi}$. Given $\bar{S}$, $\bar{\psi}$ and T, the variable $\bar{\rho}$ is determined from (62). The obtained numerical results are presented in the following table:

Measurements done with	$\bar{\psi} = 46$ $\bar{\rho} = 4.14$	$\bar{\psi} = 73$ $\bar{\rho} = 6.57$	$\bar{\psi} = 92$ $\bar{\rho} = 8.28$	$\bar{\psi} = 120$ $\bar{\rho} = 10.80$
Identification with $I = 3$	$\bar{\psi}_j = 45,97$ $\bar{\rho}_j = 4.137$	$\bar{\psi}_j = 72.82$ $\bar{\rho}_j = 6.563$	$\bar{\psi}_j = 91.58$ $\bar{\rho}_j = 8.268$	$\bar{\psi}_j = 118.97$ $\bar{\rho}_j = 10.778$
Identification with $I = 6$	$\bar{\psi}_j = 45.98$ $\bar{\rho}_j = 4.138$	$\bar{\psi}_j = 72.96$ $\bar{\rho}_j = 6.564$	$\bar{\psi}_j = 91.88$ $\bar{\rho}_j = 8.270$	$\bar{\psi}_j = 119.69$ $\bar{\rho}_j = 10.780$

Table 2: Measurements: $L_2 = 100m$, $L_1 = \sqrt{\pi}L_2$, $\gamma_1 = \gamma_2 = 10^{-2}$, $\gamma_0 = 5$, $T = 1800s$.

The analysis of the numerical results in Table 2 shows that the developed identification method leads to identify the two auxiliary variables $\bar{\psi}$ and $\bar{\rho}$ with accuracy. This latest seems to be improved by adding more measuring wells in the case where the sought $\bar{\psi}$ is far away from the initial iterate $\bar{\psi}_j = 1$. Besides, we carried out numerical experiments by considering a constant storativity function i.e., $\bar{S}(x, y) = e^{-\gamma_0}$, for all $(x, y) \in \big(0,1\big)^2$ which implies that $\bar{\rho} = 0$. The obtained results are regrouped in the following table:

Measurements done with	$\bar{\psi} = 24$	$\bar{\psi} = 81$	$\bar{\psi} = 137$
Identification with $I = 3$	$\bar{\psi}_j = 23,98$ $\bar{\rho}_j = -5.68 \times 10^{-5}$	$\bar{\psi}_j = 80.97$ $\bar{\rho}_j = -6.891 \times 10^{-6}$	$\bar{\psi}_j = 136.91$ $\bar{\rho}_j = -2.58 \times 10^{-6}$

Table 3: Measurements: $L_2 = 100m$, $L_1 = \sqrt{\pi}L_2$, $\gamma_1 = \gamma_2 = 0$, $\gamma_0 = 5$ and $T = 2400s$.

In the case of a constant storativity function, we were able to identify $\bar{\psi}$ and $\bar{\rho} = 0$ only by increasing the final monitoring time T. Indeed, starting from $T = 2400s$ the developed identification method determines with accuracy the variable $\bar{\psi}$ whereas $\bar{\rho} = 0$ is obtained with an opposite sign and relatively small values. In addition, we carried out numerical experiments in the case of a wider domain Ω. Thus, we considered the domain $\Omega = \big(0, L_1\big) \times \big(0, L_2\big)$, where $L_2 = 100m$ and $L_1 = \pi^2 L_2$. We generated measurements using $\bar{\psi} = 137$ and the storativity function $\bar{S}$ in (61) with $\gamma_1 = \gamma_2 = 10^{-2}$ and $\gamma_0 = 5$. For these experiments, we employed two different final monitoring times and studied the behaviour of the relative errors on the identified $\bar{\psi}_j$ i.e., $|\bar{\psi} - \bar{\psi}_j|/\bar{\psi}$ and on $\bar{\rho}_j$ i.e., $|\bar{\rho} - \bar{\rho}_j|/\bar{\rho}$ with respect to the used total number N_t of measurements taken during the considered monitoring time at each of the measuring wells in Table 1. The results obtained using $I = 6$ are given by:

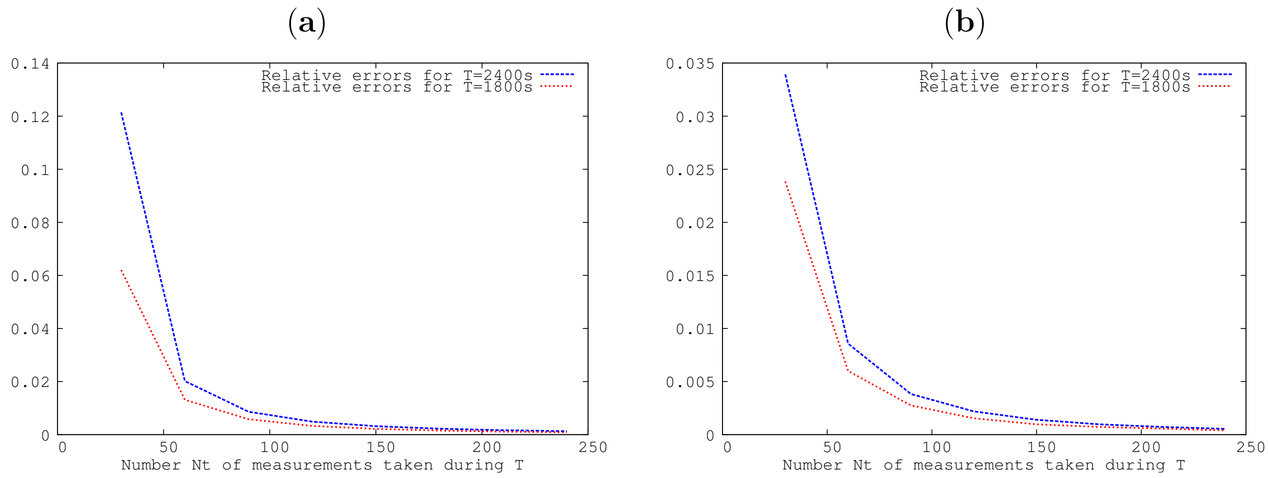

Figure 1: **(a)** Relative errors on $\bar{\psi}$ **(b)** Relative errors on $\bar{\rho}$

The behaviour of the relative errors presented in Figure 1 shows that the identified results are improved by increasing the total number N_t of measurements taken in each used measuring well during the monitoring time T. The difference between each two curves in Figure 1 observed for relatively small number N_t of measurements is due to the numerical method used to approximate the integrals with respect to the time defining the cost function. In our experiments, we used the trapezoidal rule whose the approximation error depends on the time step size $\Delta t = T/N_t$.

- **<u>Part2</u>: Identification of the storativity function S**

As far as the identification of the storativity function is concerned, we apply the algorithm of global determination developed in section 4 to identify the function of (61) that generated the used measurements i.e., $\bar{S}(x, y) = e^{-\gamma_1 L_1 x - \gamma_2 L_2 y - \gamma_0}$ in $\big(0, 1\big)^2$. We carried out numerical experiments on identifying $\bar{S}$ for different values of the coefficients γ_0, γ_1 and γ_2. The obtained results are presented in the following two tables: We give for each experiment $\big(\mathbf{E}\big)$ the values of $\gamma_0, \gamma_1, \gamma_2$ used to generate the measurements, the associated coefficients $B^{i=1,\ldots,I}$ computed from (18) and the identified storativity $\bar{S}_{ident}(x, y) = e^{g(x,y)}$.

Measurements with		B^1	B^2	B^3	B^4	B^5	B^6	$\bar{S}_{ident}(x,y) = e^{g_0}$
$(\mathbf{E}_1)$:	$\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_0 = 5$	$10^{-3} \times \big(6.67$	6.68	6.67	6.68	6.68	$6.69\big)$	$g_0 = -5.01$
$(\mathbf{E}_2)$:	$\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_0 = 10$	$10^{-5} \times \big(4.41$	4.43	4.43	4.46	4.43	$4.44\big)$	$g_0 = -10.03$

Table 4: Constant storativity: $L_2 = 100m$, $L_1 = \sqrt{\pi}L_2$, $T = 2400s$, $I = 6$ and $\bar{\psi} = 71$.

For each of the two experiments $\big(\mathbf{E}_{1,2}\big)$ presented in Table 4, the computed coefficients $B^{i=1,\dots,I}$ have about the same value. Thus, using the first step of the global determination algorithm, let $N_g = 1$ i.e., $g(x,y) = g_0$ in Ω. It follows that setting $g_0 = \ln(B^1)$ satisfies with respect to a certain tolerance all equations in (50). Therefore, we set the identified storativity function to $\bar{S}_{ident}(x,y) = e^{g_0} = B^1$. Moreover, using the results given in Table 4, we determine g_0 for the experiment $\big(\mathbf{E}_1\big)$ from $g_0 = \ln\big(6.67 \times 10^{-3}\big) = -5.01$ and for $\big(\mathbf{E}_2\big)$ from $g_0 = \ln\big(4.41 \times 10^{-5}\big) = -10.03$.

The numerical results obtained for the identification of non-constant storativity are:

Measurements with		B^1	B^2	B^3	B^4	B^5	B^6	$\bar{S}_{ident}(x,y) = e^{g(x,y)}$
$(\mathbf{E}_3)$:	$\gamma_1 = 0.03$, $\gamma_2 = 0.01$, $\gamma_0 = 5$	$10^{-4} \times \big(10.17$	8.32	10.72	7.53	11.24	$7.94\big)$	$g_1 = -3.02 \times 10^{-2}$, $g_2 = -1.01 \times 10^{-2}$, $g_0 = -4.97$
$(\mathbf{E}_4)$:	$\gamma_1 = 0$, $\gamma_2 = 0.05$, $\gamma_0 = 5$	$10^{-4} \times \big(4.51$	7.44	12.66	4.51	7.44	$2.82\big)$	$g_1 = -2.10 \times 10^{-3}$, $g_2 = -5.21 \times 10^{-2}$, $g_0 = -4.74$

Table 5: Varying storativity: $L_2 = 100m$, $L_1 = \sqrt{\pi}L_2$, $T = 1800s$, $I = 6$ and $\bar{\psi} = 71$.

Since the coefficients $B^{i=1,\dots,I}$ associated to each of the two experiments $\big(\mathbf{E}_{3,4}\big)$ presented in Table 5 don't have the same value, it follows that a polynomial g with $N_g = 1$ cann't satisfy all the equations in (50). However, as the positions of the three measuring wells b^1, b^2 and b^3 are such that the 3×3 matrix of the linear system (51) is invertible, let $N_g = 3$ i.e., $g(x,y) = g_1 L_1 x + g_2 L_2 y + g_0$. The coefficients g_0, g_1 and g_2 presented in Table 5 have been determined from solving for each experiment the linear system in (51).

6 Conclusion

We developed an identification approach that leads to reconstruct unknown storativity and transmissivity functions occurring in $2D$ groundwater equations. Using an appropriate change of variables, we transformed the groundwater equation into a diffusion-reaction one, where the diffusion term is the fraction transmissivity/storativity whereas the reaction coefficient yields the right hand side of a second order nonlinear elliptic partial differential equation satisfied by the unknown storativity function. Under some conditions on the used pumping source as well as on the number and the locations of the employed measuring wells, the developed approach starts by identifying the introduced diffusion and reaction variables. Then, it proposes local and global determination procedures for reconstructing the unknown storativity function. The unknown transmissivity is then deduced from the product of the already determined storativity and fraction transmissivity/storativity functions. The numerical results carried out in this paper show that the developed approach determines accurately the used storativity and fraction functions.

References

[1] Abdel-Aziz H., Abdel-Gawad Ahmed and El-Hadi H.A. (2009) Parameter estimation of pumping test data using genetic algorithm, Thirteenth International Water Technology Conference, IWTC 13, Hurghada-Egypt.

[2] Alarcon S., Garcia-Mellian J., Quaas A. (2012) Keller-Osserman type conditions for some elliptic problems with gradient terms, J. Diff. Equations 252:2, pp. 886-914.

[3] Anderson M.P. and Woessner W.W. (1992) Applied Groundwater Modeling, Academic Press, Inc., San Diego, CA.

[4] Bear J. (1979) Hydraulics of Groundwater, McGraw-Hill, New York.

[5] Beckie R. and Harvey C.F. (2002) What does a slug test measure: an investigation of instrument response and the effects of heterogeneity, Water Resour. Res. 38:12, pp: 1-14.

[6] Berestycki H., Hamel F. and Nadirashvili N. (2005) Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena, Commun. Math. Phys. 253, pp. 451-480.

[7] Bulter J. (2009) Pumping tests for aquifer evaluation, time for change?, Groundwater 46, pp. 615-617.

[8] Castagna M. and Bellin A. (2009) A Bayesianapproach for inversion of hydraulic tomographic data, Water Resources Research, Vol. 45, W04410, DOI: 10.1029/2008WR007078.

[9] Chen X.F., Lou Y. (2008) Principal eigenvalue and eigenfunction of an elliptic operator with large advection and its application to a competition model, Indiana Univ. Math. J. 57, pp. 627-658.

[10] Cleveland T.G. (1996) Type-Curve Matching Using a Computer Spreadsheet, Groundwater, 34:3, pp. 554-562.

[11] Cooper H.H. and Jacob C.E. (1946) A generalized graphical method for evaluating formation constants and summarizing well-field history, American Geophysical Union Transactions, 27:4, pp. 526-534.

[12] El Badia A. and Ha Duong T. (2002) On an inverse source problem for the heat equation: Application to a pollution detection problem, Journal of Inverse Ill-posed Problems, 10, pp. 585-599.

[13] El Jai A. and Pritchard G. (1986) Capteurs et Actionneurs dans l'Analyse des systèmes distribués (RMA, 3) Paris: Masson.

[14] Fursikov A. and Imanuvilov O. Y. (1996) Controllability of evolution equations, Lecture Notes, Research Institute of Mathematics, Seoul National University, Korea.

[15] Garca G., Osses A., Puel J.-P. (2011) A null controllability data assimilation methodology applied to a large scale ocean circulation model, *M2AN Math. Model. Numer. Anal.*, 45:2, pp. 361-86.

[16] Jim Yeh T.C. (1992) Stochastic Modelling of Groundwater Flow and Solute Transport in Aquifers, Hydrlogical Processes, 6, pp. 369-395.

[17] Hamdi Adel (2016) Detection and identification of multiple unknown time-dependent point sources occurring in $1D$ evolutionj transport equations, Inverse Problems in Science and Engineering, 25:4, pp. 532-554.

[18] Hoeksema R.J. and Kitanidis P.K. (1984) An application of the geostatistical approach to the inverse problem in two-dimensional groundwater modeling, Water Resour. Res., 20, pp. 1003-1020.

[19] Lin F. H. (1990) A uniqueness theorem for parabolic equations, Comm. Pure Appl. Math., 43, pp. 125-136.

[20] Lions J.L. (1992) Pointwise control for Distributed Systems in Control and Estimation in distributed ParametersSystems, Edited by Banks H.T. SIAM.

[21] Meier P.M., Carrera J. and Sanchez-Villa X. (1999) A numerical study on the relationship between transmissivity and specific capacity in heterogeneous aquifers, Groundwater 37:4, pp:611-617.

[22] Nelson R. W. (1960) In-place measurement of permeability in heterogeneous media: Experimental and computational considerations, J. Geophys. Res. 66, pp. 2469-2478.

[23] Radulescu V. (2007) Singular phenomena in nonlinear elliptic problems: from boundary blow-up solutions to equations with singular nonlinearities, Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 4 (Michel Chipot), pp. 483-591.

[24] Richter G. R. (1981) An inverse problem for the steady state diffusion equation, SIAM J. Math. Anal. 41, pp. 210-221.

[25] Rogelio V. G., Mauro G., Giansilvio P. and Guido P. (1997) The differential system method for the identification of transmissivity and storativity, Transport in Porous Media, 26, pp. 339-371.

[26] Shapoori V., Peterson T.J., Western A.W. and Costelloe (2015) Estimating aquifer properties using groundwater hydrograph modelling, Hydrol. Process., DOI:10.1002/hyp.10583.

[27] Schwartz L. (1966) Théorie des distributions, Hermann-Paris.

[28] Shapoori V., Peterson T.J., Western A.W. and Costelloe J.F. (2015) Estimating aquifer properties using groundwater hydrgraph modelling, Hydrological Processes, DOI: 10.1002/hyp.10583.

[29] Sterrett R.J. (2007) Groundwater and wells, 3rd Eddition. New Brighton, Minnesota: Johnson Screens.

[30] Stewart B. (1980) Generation of analytic semigroups by strongly elliptic operators under general boundary conditions, American Mathematical Society, 259, pp. 299-310.

[31] Titchmarsh, E. C. (1939) Introducton to the theory of Fourier Integrals, Oxford.

[32] Todd D.K. (2004) Groundwater Hydrology. 2nd Edn., John Wiley, New-York.

[33] Theis C.V. (1935) The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using groundwater storage, American Geophysical Union Transaction, 16, pp. 519-524.

[34] Yeh T.J. and Lee C. (2007) Time to change the way we collect and analyze data for aquifer characterization, Groundwater 45:116-118.