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Abstract. In this paper, a new robust observer based sliding mode control approach is proposed for uncertain linear discrete-time delay system. The system under study is subject to time-varying states delay, and to different types of disturbances such as unmatched Unknown but norm-Bounded (UBB) parametric uncertainties and external disturbances. Additional nonlinearities have been considered as well. Such system nonlinearities are supposed to verify a Hölder condition of order α. A new delay-dependent sufficient conditions ensuring the robust asymptotic stability of the closed-loop system is derived in terms of Linear Matrix Inequalities (LMI). With regards to these conditions and considering the novel sliding surface-based observer, a new discrete-time sliding mode control law is designed. Finally, an illustrative example is exhibited to show the validity of the proposed control scheme.

Keywords: Observer-based Discrete-Time Sliding Mode Control, time-varying delay, Hölder nonlinearities, robustness

1 Introduction

Time-delay is frequently encountered in many real systems including different industrial applications (see [3],[10] and the references therein). However, its existence may cause instability and oscillations which results in system performances degradation [2]. Moreover, the presence of the uncertainties, in Time-Delay Systems (TDS), may dwindle further the system performances. Therefore, in order to improve the practical applications in industrial processes, it is tremendously important to focus on the stability analysis and the control design methods for uncertain TDS.
Among these control approaches, several works have focused on the state-feedback control methods design, whereas, in such case it is required that the system states are available for measurements, an assumption which is, from a practical point of view, seldom verified. Consequently, instead of designing state-feedback controllers, attractive attentions have been devoted to the design of output-feedback controllers i.e observer-based controllers [11] as the Sliding Mode Observers (SMO) which have been widely adopted in practical industrial systems. In [8] the author underlined the advantages of

SMO, mainly its ability to generate a sliding motion on the error between the measured plant output and the observer output. Therefore, the SMO ensures the convergence in finite time of the estimated states toward the real system states. Due to these features sliding-mode observers have attracted considerable interests, in the past few years, and several design methods have been proposed for linear uncertain TDS.
Over and above that, and since the most of real systems are affected by disturbances, therefore a crucial problem that ought to be tackled is how to deal with the disturbances in the SMO design [5] and the control design as well. The presence of Hölder nonlinearities, which are particular case of such disturbances, still is a challenging problem in robust control design [12] and so does in SMO design too [6].
More recently, the design of SMO for uncertain linear time-delay systems under Hölder nonlinearities has received attractive attention, even though, the design of such observer in the discrete-time domain is still much less mature compared to the continuous-time domain. It is important to underline, however, that the most of sliding mode observers designs are delay-independent which increases, in fact, the design conservatism. Regarding all the above points, the aim of this paper is, therefore, to design a new SMO under less conservative conditions for uncertain linear discrete time-delay systems subject to Hölder nonlinearities.
The design of robust observer based controller depends, on the one hand, on the considered class of system, and on the other hand, on the control method adopted for the design. In fact, the author in [7] has designed an observer for an uncertain linear discrete-time system subject to unknown bounded time-varying delay on state, time-varying parametric uncertainties and unknown disturbance. The considered system in [7] is also affected by nonlinearity, which is a single variable vector-valued function, and is required to verify a local Lipschitz condition. The proposed observer structure does not involve neither the delayed term $A_d x(k-d(k))$ nor the term multiplying the input $Bu(k)$. The observer design has been achieved based on the H $^\infty$ criterion. In [4], the author synthesized an observer for a linear discrete singular system subject to known time-varying state delay, UBB parameter uncertainties and exogenous disturbance. The matrix multiplying the exogenous disturbance is certain. Such system does not incorporate any Hölder nonlinearity. The sliding mode control is the method whereby the observer is designed. In [12] the observer is designed for a class of system similar to the one examined in the present work. However we have to precise that our proposed observer is distinguished from the one presented in [12], in fact, in this last reference the observer structure does not contain the delayed term $A_d x(k-d(k))$, besides, H^∞ control is the control method by which the observer design is realized.
Considering the above discussions, the problem investigated in our current paper is slightly more difficult than these examined in the cited references, the observer structures developed in these works are, thus, no longer suitable. The main goals of this paper are listed as follows:

- The design of a new robust observer-based SMC for uncertain linear discrete-time delay systems. The studied system is affected by time-varying bounded delay on the states, UBB uncertainties on all parameters, unmatched external disturbances and it is also subject to Hölder nonlinearities. The proposed observer is able to estimate and to reconstruct the system states inspite of the features of the system.

- The design of a novel sliding surface-based observer. The stability of the proposed sliding surface is conducted by means of a suitable choice of a Lypunov Krasovski Functional. A new less conservative delay-dependent sufficient condition guaranteeing the robust asymptotic stability of the closed-loop system is generated in terms of LMIs.
- The synthesis of a new sliding mode control law.

This paper is organised as follows: System description and the preliminaries are given in Section 2. The new robust observer-based discrete-time sliding mode control design is presented in Section 3. Such section is divided into three parts: the first one is devoted to the observer design, the second section addresses the synthesis of the novel sliding surface while the third part deals with the synthesis of the new control law. A numerical example which elucidates the validity of our proposed control approach is provided in Section 4.

2 System description and preliminaries

Consider the following uncertain discrete time-delay system described by:

$$\begin{cases} x(k+1) = (A+\Delta A)\,x(k) + (A_d+\Delta A_d)\,x(k-d(k)) + Bu(k) + (D+\Delta D)\,w(k) \\ \qquad\qquad + (R+\Delta R)\,f(x(k), x(k-d(k))) \\ y(k) = C_y x(k) \\ x(k) = \psi(k) \;\; \forall \;\; k = -d_M, -d_M+1, \cdots, 0 \end{cases} \tag{1}$$

Where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^r$ is the measured output, $w \in \mathbb{R}^n$ is the exogenous disturbance signal assumed to belong to $l_2[0,\infty)$, f is the system nonlinearities required to satisfy a hölder condition. ψ is a given initial state condition. $d(k)$ is the time-varying delay affecting the system states, it is assumed to be bounded as: $0 \leq d_m \leq d(k) \leq d_M$. The matrices A, A_d, B, D, R, C_y are real constant matrices of appropriate dimensions. The parameter uncertainties ΔA, ΔA_d, ΔD and ΔR are assumed to be Unknown But Bounded (UBB) and are written with regard to the following form:

$$[\Delta A(k) \;\; \Delta A_d(k) \;\; \Delta D(k) \;\; \Delta R(k)] = MF(k)\,[N_1 \;\; N_2 \;\; N_3 \;\; N_4] \tag{2}$$

Where N_1, N_2, N_3 and N_4 are known real constant matrices of appropriate dimensions, while $F(k)$ is an unknown real and possible time-varying matrix with Lebesgue measurable elements satisfying $F^T(k)F(k) \leq I \;\; \forall \;\; k$
In what follows and regarding system (1), we give the preliminaries required for the statment of the main results:

Proposition 1. *The pairs (A,B) and (A,C_y) are controllable and observable respectively.*

Proposition 2. *The matrix C_y has a full row rank r with $r \leq n$. It implies the existence of a right inverse matrix $C_{yL}^{-1} \in \mathbb{R}^{n\times r}$ such that $C_y C_{yL}^{-1} = I_r$.*

Proposition 3. *The nonlinearity $f(x(k), x(k-d(k))) : \mathbb{R}^n \times R^n \mapsto \mathbb{R}^n$ in system (1) is a multivariable vector-valued function of the following form*

$$f(x(k), x(k-d(k))) = \begin{bmatrix} f_1(x_1(k), x_1(k-d(k))) \\ f_2(x_2(k), x_2(k-d(k))) \\ \vdots \\ f_n(x_n(k), x_n(k-d(k))) \end{bmatrix} \tag{3}$$

Where $f_i(x_i(k), x_i(k-d(k))) : \mathbb{R}^2 \mapsto \mathbb{R}$ are multivariable real-valued functions, representing the i-th component of $f(\cdot,\cdot)$, with $\forall\ i = (1,\ldots,n)$. Moreover, for k_1 and k_2, the function (3) has to satisfy the finite α Hölder condition expressed as follows:

$$\begin{aligned} &d_Y(f(x_i(k_1), x_i(k_1-d(k_1))), f(x_i(k_2), x_i(k_2-d(k_1)))) \\ &\leq g\left(\left|d_X(x_i(k_1), x_i(k_2))\right|^{\alpha} + \ + \left|d_X(x_i(k_1-d(k_1)), x_i(k_2-d(k_2)))\right|^{\alpha}\right) \end{aligned} \tag{4}$$

Where α, which denotes the exponent of the Hölder condition, and the term g are known real constants. $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$, are respectively the metric on the sets X and Y, and are defined as:

$$\begin{cases} d_Y\left(f(x_i(k_1), x_i(k_1-d(k_1))), f(x_i(k_2), x_i(k_2-d(k_1)))\right) \\ = \left|f(x_i(k_1), x_i(k_1-d(k_1))) - f(x_i(k_2), x_i(k_2-d(k_2)))\right| \\ d_X(x_i(k_1), x_i(k_2)) = x_i(k_1) - x_i(k_2) \ \ and \\ d_X(x_i(k_1-d(k_1)), x_i(k_2-d(k_2))) = x_i(k_1-d(k_1)) - x_i(k_2-d(k_2)) \end{cases} \tag{5}$$

Proposition 4. *The nonlinearity term $f(x(k), x(k-d(k))) : \mathbb{R}^n \times R^n \mapsto \mathbb{R}^n$ in system (1) is said to belong to the sector $[\Lambda_1, \Lambda_2]$ if the condition (6) holds $\forall\ x(k_1),\ x(k_2) \in \mathbb{R}^n$ and their exits two known real constant matrices of appropriate dimensions Λ_1 and Λ_2 such that $\Lambda_1 - \Lambda_2$ is a positive definite symmetric matrice*

$$\begin{aligned} &\left[f(x(k_1), x(k_1-d(k_1))) - f(x(k_2), x(k_2-d(k_2))) - \ -\Lambda_1(x(k_1) - x(k_1))\right]^T \\ &\left[f(x(k_1), x(k_1-d(k_1))) \ -f(x(k_2), x(k_2-d(k_2))) - \Lambda_2(x(k_1) - x(k_1))\right] \leq 0 \end{aligned} \tag{6}$$

Let us remind also the Lemma 1 which will be used in the proof of Thoerem 2.

Lemma 1. *[1] Given matrices Y, D, E of appropriate dimensions where Y symmetric, then $Y + DF(k)E + E^T F^T(k) D^T < 0$ for all $F(k)$ satisfying $F^T(k)F(k) \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that $Y + \varepsilon DD^T + \varepsilon^{-1} EE^T < 0$*

3 New robust observer-based discrete-time sliding mode control (DSMC) design

3.1 Observer design

Let us consider the following observer structure:

$$\begin{cases} \widehat{x}(k+1) = A\widehat{x}(k) + A_d\widehat{x}(k-d(k)) + Bu(k) + Rf(\widehat{x}(k), \widehat{x}(k-d(k))) + L[y(k) - \widehat{y}(k)] \\ \widehat{y}(k) = C_y\widehat{x}(k) \\ \widehat{x}(k) = \psi_1(k) \ \ \forall\ k = -d_M, -d_M+1, \cdots, 0 \end{cases} \tag{7}$$

Where $\widehat{x} \in \mathbb{R}^n$ represents the estimation of the state x, $\widehat{y} \in \mathbb{R}^r$ denotes the observer output, ψ_1 is the observer initial condition and $L \in \mathbb{R}^{n\times r}$ is the observer gain to be designed later. We define the state estimated error as follows:

$$e(k) = x(k) - \widehat{x}(k) \tag{8}$$

With few simple steps we can deduce the expression of the system error dynamics:

$$\begin{aligned} e(k+1) = {} & [A+\Delta A - LC_y]\, e(k) + (A_d + \Delta A_d)\, e(k-d(k)) + \Delta A\widehat{x}(k) + \Delta A_d \widehat{x}(k-d(k)) \\ & + (D+\Delta D)\, w(k) + (R+\Delta R)\, f(x(k), x(k-d(k))) - Rf(\widehat{x}(k), \widehat{x}(k-d(k))) \end{aligned} \tag{9}$$

3.2 Sliding surface design

Inspired from [4] and [12], we consider the novel sliding surface given as follows:

$$S(k) = G\widehat{x}(k) - G(A+BK)\widehat{x}(k-1) \tag{10}$$

Where $G \in \mathbb{R}^{m\times n}$ is chosen such that (GB) is nonsingular and $K \in \mathbb{R}^{m\times n}$ is the controller gain matrix to be designed. Following the definition of this sliding surface then the expression of the equivalent input u_{eq} is deduced:

$$u_{eq}(k) = -(GB)^{-1} G\left[A_d\widehat{x}(k-d(k)) + Rf(\widehat{x}(k), \widehat{x}(k-d(k))) + LC_y e(k)\right] + K\widehat{x}(k) \tag{11}$$

Substituting (11) into (7) we get:

$$\begin{aligned} \widehat{x}(k+1) = {} & [A+BK]\widehat{x}(k) + \left[I_n - B(GB)^{-1}G\right] A_d \widehat{x}(k-d(k)) + \left[I_n - B(GB)^{-1}G\right] LC_y e(k) \\ & + \left[I_n - B(GB)^{-1}G\right] Rf(\widehat{x}(k), \widehat{x}(k-d(k))) \end{aligned} \tag{12}$$

Let us define the augmented state vector as $\overline{x}(k) = \begin{bmatrix} e(k) \\ \widehat{x}(k) \end{bmatrix}$, Now by exploiting (9) and (12) then we can write:

$$\overline{x}(k+1) = \boldsymbol{\mathcal{A}}\overline{x}(k) + \boldsymbol{\mathcal{A}}_d \overline{x}(k-d(k)) + \boldsymbol{\mathcal{D}} w(k) + \boldsymbol{\mathcal{R}}\,\overline{\boldsymbol{\mathcal{F}}}_k \tag{13}$$

where

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} [A+\Delta A - LC_y] & \Delta A \\ \left[I_n - B(GB)^{-1}G\right] LC_y & [A+BK] \end{bmatrix}, \quad \boldsymbol{\mathcal{A}}_d = \begin{bmatrix} (A_d + \Delta A_d) & \Delta A_d \\ 0 & \left[I_n - B(GB)^{-1}G\right] A_d \end{bmatrix},$$

$$\boldsymbol{\mathcal{R}} = \begin{bmatrix} (R+\Delta R) & -R \\ 0 & \left[I_n - B(GB)^{-1}G\right] R \end{bmatrix}, \quad \boldsymbol{\mathcal{D}} = \begin{bmatrix} (D+\Delta D) \\ 0 \end{bmatrix}, \overline{\boldsymbol{\mathcal{F}}}_k = \begin{bmatrix} f(x(k), x(k-d(k))) \\ f(\widehat{x}(k), \widehat{x}(k-d(k))) \end{bmatrix} \tag{14}$$

Theorem 1. *Consider the proposed sliding surface (10), and the designed observer (7), then the closed-loop augmented system (13) is asymptotically stable if the inequaliy (15) holds*

$$\Phi = \begin{bmatrix} \mathcal{A}^T P\mathcal{A} - P + (d_M - d_m + 1)Q & \mathcal{A}^T P\mathcal{A}_d & \mathcal{A}^T P\mathcal{D} & \mathcal{A}^T P\mathcal{R} \\ * & \mathcal{A}_d^T P\mathcal{A}_d - Q & \mathcal{A}_d^T P\mathcal{D} & \mathcal{A}_d^T P\mathcal{R} \\ * & * & \mathcal{D}^T P\mathcal{D} & \mathcal{D}^T P\mathcal{R} \\ * & * & * & \mathcal{R}^T P\mathcal{R} \end{bmatrix} < 0 \quad (15)$$

where P and Q are positive definite matrices $\in \mathbb{R}^{2n\times 2n}$ to be designed.

Proof. Let consider the following Lyapunov-Krasovskii functional candidate inspired from [12]

$$V(k) = V_1(k) + V_2(k) + V_3(k)$$

where

$$\begin{aligned} V_1(k) &= \overline{x}^T(k)P\overline{x}(k);\; V_2(k) = \sum_{i=k-d(k)}^{k-1} \overline{x}^T(i)Q\overline{x}(i) \\ V_3(k) &= \sum_{j=-d_M+2}^{-d_m+1} \sum_{i=k+j-1}^{k-1} \overline{x}^T(i)Q\overline{x}(i) \end{aligned} \quad (16)$$

To prove the Asymptotic stability of the closed-loop augmented system (13) we have to demonstrate that the increment $\Delta V(k) = V(k+1) - V(k)$ is negative definite. To do so, we compute each of the increment ΔV_i, $(i = 1,2,3,4)$.
Through simple manipulations and using (13) and (16), we finally obtain

$$\Delta V(k) \leq \begin{bmatrix} \overline{x}(k) \\ \overline{x}(k-d(k)) \\ w(k) \\ \overline{\mathcal{F}}_k \end{bmatrix}^T \Phi \begin{bmatrix} \overline{x}(k) \\ \overline{x}(k-d(k)) \\ w(k) \\ \overline{\mathcal{F}}_k \end{bmatrix} \quad (17)$$

where Φ is defined in (15). According to (17), ΔV is definite negative if $\Phi < 0$ which satisfied the condition stated in Theorem 1. □

Regarding (10) then the complete determination of the sliding surface requires the determination of the observer gain matrix L and the control gain matrix K as well, to this end, in what follows, we give Theorem 2 which reformulates the results of Theorem 1 by LMI approach.

Theorem 2. *Consider the proposed sliding surface (10), and the designed observer (7). The closed-loop augmented system (13) is asymptotically stable for all known time varying bounded delay $d(k)$, if there exist scalars $\varepsilon > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$, positive definite symmetric matrices W_i, $(i = 1,2,3,4) \in \mathbb{R}^{n\times n}$, matrices $L_0 \in \mathbb{R}^{n\times n}$ and $K_0 \in$*

$\mathbb{R}^{n\times m}$ *such that the LMI (18) holds*

$$\begin{bmatrix} -W_1 & 0 & 0 & 0 & 0 & 0 & 0 & W_1A^T-L_0 & L_0\overline{G}^T & W_1N_1^T & 0 & W_1 & 0 \\ * & -W_2 & 0 & 0 & 0 & 0 & 0 & 0 & W_2A^T+K_0B^T & W_2N_1^T & 0 & 0 & W_2 \\ * & * & -W_3 & 0 & 0 & 0 & 0 & W_3A_d^T & 0 & W_3N_2^T & 0 & 0 & 0 \\ * & * & * & -W_4 & 0 & 0 & 0 & 0 & W_4\left(\overline{G}A_d\right)^T & W_4N_2^T & 0 & 0 & 0 \\ * & * & * & * & -(\lambda_1+\lambda_2)\widetilde{V}_1 & -\lambda_1\widetilde{V}_2 & -\lambda_2\widetilde{V}_2 & D^T & 0 & N_3^T & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda_1 I & 0 & R^T & 0 & N_4^T & 0 & 0 & 0 \\ * & * & * & * & * & * & -\lambda_2 I & -R^T & \left(\overline{G}R\right)^T & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -W_1 & 0 & 0 & \varepsilon M & 0 & 0 \\ * & * & * & * & * & * & * & * & -W_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & \frac{-W_3}{\overline{d}} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & \frac{-W_4}{\overline{d}} \end{bmatrix} < 0 \tag{18}$$

and where $\overline{d} = (d_M - d_m + 1)$, $\overline{G} = \left[I_n - B(GB)^{-1}G\right]$ *Moreover, the expression of the controller gain K and the observer gain L are :*

$$K = K_0^T\left(W_2^{-1}\right)^T, \ L = L_0^T\left(W_1^{-1}\right)^T C_{yL}^{-1} \ \text{and} \ C_yC_{yL}^{-1} = I_r \tag{19}$$

Proof. Let us define the vector $\xi(k)$ as:

$$\xi(k) = \left[\overline{x}^T(k)\ \overline{x}^T(k-d(k))\ w^T(k)\ \overline{\mathcal{F}^T}_k\right]^T \tag{20}$$

Where $\overline{x}(k) = \begin{bmatrix} e(k) \\ \widehat{x}(k)\end{bmatrix}$, $\overline{x}(k-d(k)) = \begin{bmatrix} e(k-d(k)) \\ \widehat{x}(k-d(k))\end{bmatrix}$, $\overline{\mathcal{F}}_k = \begin{bmatrix} f(x(k), x(k-d(k))) \\ f(\widehat{x}(k), \widehat{x}(k-d(k)))\end{bmatrix}$.

Let us consider the two matrices P and Q such that:

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2\end{bmatrix}, \text{ where} P_1,\ P_2,\ Q_1,\ Q_2 \in \mathbb{R}^{n\times n}$$

Using (20) and (17), one can show that:

$$\Delta V(k) \le \xi^T(k)\Phi_1\xi(k) + \xi^T(k)\left[\mathcal{A}\ \mathcal{A}_d\ \mathcal{D}\ \mathcal{R}\right]^T P\left[\mathcal{A}\ \mathcal{A}_d\ \mathcal{D}\ \mathcal{R}\right]\xi(k) < 0 \tag{21}$$

Where:

$$\Phi_1 = \begin{bmatrix} -P_1+(d_M-d_m+1)Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -P_2+(d_M-d_m+1)Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix} \quad (22)$$

Following the same procedure as in [12] and with respect to Proposition 4 and (6) we obtain:

$$\begin{cases} \begin{bmatrix} w(k) \\ f(x(k),x(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \widetilde{V}_1 & \widetilde{V}_2 \\ \widetilde{V}_2^T & I \end{bmatrix} \begin{bmatrix} w(k) \\ f(x(k),x(k-d(k))) \end{bmatrix} \leq 0 \\ \begin{bmatrix} w(k) \\ f(\widehat{x}(k),\widehat{x}(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \widetilde{V}_1 & \widetilde{V}_2 \\ \widetilde{V}_2^T & I \end{bmatrix} \begin{bmatrix} w(k) \\ f(\widehat{x}(k),\widehat{x}(k-d(k))) \end{bmatrix} \leq 0 \end{cases} \quad (23)$$

Where $\widetilde{V}_1$ and $\widetilde{V}_2$ are defined as follows:

$$\widetilde{V}_1 = \left(\Lambda_1^T\Lambda_2+\Lambda_2^T\Lambda_1\right)/2 \text{ and } \widetilde{V}_2 = -\left(\Lambda_1^T+\Lambda_2^T\right)/2 \quad (24)$$

Using (20), (21) and (23) we can deduce that there exists two positive scalars λ_1, λ_2 such that the following inequality holds

$$\Delta V(k) \leq \xi^T(k)\Phi_2\xi(k) + \xi^T(k)\left[\mathcal{A}\ \mathcal{A}_d\ \mathcal{D}\ \mathcal{R}\right]^T P\left[\mathcal{A}\ \mathcal{A}_d\ \mathcal{D}\ \mathcal{R}\right]\xi(k) < 0 \quad (25)$$

Where

$$\Phi_2 = \begin{bmatrix} -P_1+(d_M-d_m+1)Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -P_2+(d_M-d_m+1)Q_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 & 0 \\ * & * & * & * & -\lambda_1\widetilde{V}_1-\lambda_2\widetilde{V}_1 & -\lambda_1\widetilde{V}_2 & -\lambda_2\widetilde{V}_2 \\ * & * & * & * & * & -\lambda_1 I & 0 \\ * & * & * & * & * & * & -\lambda_2 I \end{bmatrix} \quad (26)$$

From (25) we have

$$\xi^T(k)\left\{\Phi_2+\left[\mathcal{A}\ \mathcal{A}_d\ \mathcal{D}\ \mathcal{R}\right]^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \left[\mathcal{A}\ \mathcal{A}_d\ \mathcal{D}\ \mathcal{R}\right]\right\}\xi(k) < 0 \quad (27)$$

Applying the schur complement to (27), then in the resulting matrix inequality by employing (14) we replace, $\mathcal{A}$, $\mathcal{A}_d$, $\mathcal{D}$, $\mathcal{R}$ by their expressions. We take into account (26) and (2) and we define $\widetilde{M}$,$\widetilde{N}$ and Φ_3 as in (28) and where $\overline{d} = (d_M - d_m + 1)$, and

$\overline{G} = \left[I_n - B(GB)^{-1}G\right].$

$\widetilde{M} = \left[0\;0\;0\;0\;0\;0\;0\;M^T\;0\right]^T,\ \widetilde{N} = \left[N_1\;N_1\;N_2\;N_2\;N_3\;N_4\;0\;0\;0\right]$

$$\Phi_3 = \begin{bmatrix} -P_1+\overline{d}Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & [A-LC_y]^T & \left(\overline{G}LC_y\right)^T \\ * & -P_2+\overline{d}Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & [A+BK]^T \\ * & * & -Q_1 & 0 & 0 & 0 & 0 & A_d^T & 0 \\ * & * & * & -Q_2 & 0 & 0 & 0 & 0 & \left(\overline{G}A_d\right)^T \\ * & * & * & * & -(\lambda_1+\lambda_2)\widetilde{V}_1 & -\lambda_1\widetilde{V}_2 & -\lambda_2\widetilde{V}_2 & D^T & 0 \\ * & * & * & * & * & -\lambda_1 I & 0 & R^T & 0 \\ * & * & * & * & * & * & -\lambda_2 I & -R^T & \left(\overline{G}R\right)^T \\ * & * & * & * & * & * & * & -P_1^{-1} & 0 \\ * & * & * & * & * & * & * & * & -P_2^{-1} \end{bmatrix} \quad (28)$$

By simple manipulations one can show the following matrice inequality:

$$\Phi_3 + \widetilde{M}F(k)\widetilde{N} + \left(\widetilde{M}F(k)\widetilde{N}\right)^T < 0 \quad (29)$$

By means of Lemma 1, the matrix inequality (29) is equivalent to

$$\Phi_3 + \varepsilon\widetilde{M}\widetilde{M}^T + \varepsilon^{-1}\widetilde{N}^T\widetilde{N} < 0 \quad (30)$$

Applying the schur complement to (30), we get

$$\begin{bmatrix} \Phi_3 & \widetilde{N}^T & \varepsilon\widetilde{M} \\ \widetilde{N} & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0 \quad (31)$$

In (31), we substitute Φ_3, $\widetilde{M}$ and $\widetilde{N}$ by (28) then we Pre and post-multiply the resulting matrix inequality by

$$\Omega = diag\left(P_1^{-1}, P_2^{-1}, Q_1^{-1}, Q_2^{-1}, I, I, I, I, I, I\right)$$

By applying twice a time the schur complement to the resulting matrix inequality then we obtain (18) and where

$$\begin{gathered} W_1 = P_1^{-1},\ W_2 = P_2^{-1},\ W_3 = Q_1^{-1},\ W_4 = Q_2^{-1} \\ K = K_0^T\left(W_2^{-1}\right)^T \text{ and } L = L_0^T\left(W_1^{-1}\right)^T C_{yL}^{-1} \end{gathered} \quad (32)$$

□

3.3 Synthesis of the new control law

Theorem 3. *Given the proposed sliding surface (10) and the proposed observer (7), by considering the new control law (33), then the closed-loop augmented system (13) is asymptotically stable in quasi-sliding mode (QSM) for any time varying bounded delay* $d(k)$*. In addition, the trajectory of the closed-loop augmented system (13) are driven to QSM despite the presence of uncertainties, nonlinearity and time varying bounded delay.*

$$\begin{aligned} u(k) = {} & u(k-1) - \xi S(k) - \beta \exp^{-\mu k} sign(S(k)) - (GB)^{-1} GR\gamma(k) \\ & + K[\widehat{x}(k) - \widehat{x}(k-1)] - (\omega_1 - \omega_2)(GB)^{-1} \times \left\{(1,\dots,1)^T\right\}_{m\times 1} \\ & - (GB)^{-1} GA_d [\widehat{x}(k-d(k)) - \widehat{x}(k-1-d(k-1))] \end{aligned} \tag{33}$$

where $\gamma(k)$ *is defined in (34),* ω_1 *and* ω_2 *are positive scalars given in (35), and* ξ*,* μ *and* β *are the parameters of the reaching condition verifying* $0 < \xi < 1$, $\mu > 0$, $\beta > 0$.

$$\gamma(k) = g \;\; diag(1,\dots,1)_{n\times n} \left\{ |\widehat{x}(k) - \widehat{x}(k-1)|^{\alpha} + |\widehat{x}(k-d(k)) - \widehat{x}(k-1-d(k-1))|^{\alpha} \right\} \tag{34}$$

$$\begin{cases} \omega_1 = \|GLy(k)\| + \|GL\widehat{y}(k)\| \, diag(1,\dots,1)_{m\times m} \\ \omega_2 = \|GLy(k-1)\| + \|GL\widehat{y}(k-1)\| \, diag(1,\dots,1)_{m\times m} \end{cases} \tag{35}$$

Proof. Let us consider the following reaching law [4]:

$$\begin{cases} \Delta S(k) \le -\xi S(k) - \beta \exp^{-\mu k} sign(S(k)) & \text{if } S(k) > 0 \\ \qquad\;\; \ge -\xi S(k) - \beta \exp^{-\mu k} sign(S(k)) & \text{if } S(k) < 0 \end{cases} \tag{36}$$

Using (7) and (10) then we can write:

$$\begin{aligned} \Delta S(k) = {} & GA_d \widehat{x}(k-d(k)) - GA_d \widehat{x}(k-1-d(k-1)) + GLC_y[x(k) - \widehat{x}(k)] \\ & - GLC_y[x(k-1) - \widehat{x}(k-1)] - GBK[\widehat{x}(k) - \widehat{x}(k-1)] + GB[u(k) - u(k-1)] \\ & + GR\left[f(\widehat{x}(k), \widehat{x}(k-d(k))) - f(\widehat{x}(k-1), \widehat{x}(k-1-d(k-1)))\right] \end{aligned} \tag{37}$$

With respect to Proposition 3, using (34) and (37) then one can show:

$$\begin{aligned} \Delta S(k) \le {} & GA_d \widehat{x}(k-d(k)) - GA_d \widehat{x}(k-1-d(k-1)) + GLC_y[x(k) - \widehat{x}(k)] \\ & + GR\gamma(k) + GB[u(k) - u(k-1)] - GLC_y[x(k-1) - \widehat{x}(k-1)] - GBK[\widehat{x}(k) - \widehat{x}(k-1)] \end{aligned} \tag{38}$$

Considering (35) we can deduce:

$$\begin{cases} -\omega_1 \left\{(1,\dots,1)^T\right\}_{m\times 1} \le GLy(k) - GL\widehat{y}(k) \le \omega_1 \left\{(1,\dots,1)^T\right\}_{m\times 1} \\ -\omega_2 \left\{(1,\dots,1)^T\right\}_{m\times 1} \le GLy(k-1) - GL\widehat{y}(k-1) \le \omega_2 \left\{(1,\dots,1)^T\right\}_{m\times 1} \end{cases} \tag{39}$$

Refering to (36), (38) and (39), we finally obtain the new control law (33). □

4 Numerical example

In the present section we endeavour to demonstrate the effectiveness of the proposed control scheme. To do so, in the following we will report simulation results that are obtained from the application of the proposed controller (33) to the system (1) which is totally defined by considering the following parameters:

$$A=\begin{bmatrix}-0.0912 & 0.059\\ 0.0570 & -0.1020\end{bmatrix}, A_d=\begin{bmatrix}-1 & -0.52\\ 0.52 & -0.01\end{bmatrix}, D=\begin{bmatrix}0.176 & 0.7176\\ 0.1 & 0.421\end{bmatrix}, R=\begin{bmatrix}0.01 & 0\\ 0.01 & 0.01\end{bmatrix}$$
$$B=\begin{bmatrix}1.39\\ -1.3\end{bmatrix}, C_y=\begin{bmatrix}0.126 & -0.1019\end{bmatrix} \text{ with SVD technique } C_{yL}^{-1}=\begin{bmatrix}4.7982\\ -3.8805\end{bmatrix}$$
$$M=\begin{bmatrix}0.3 & 0.2\\ 0.1 & 0.8\end{bmatrix}, F(k)=\begin{bmatrix}\sin(2\pi kT_s) & 0\\ 0 & 0.3\sin(k\pi T_s)\end{bmatrix}, N_1=\begin{bmatrix}0.1 & 0.4\\ 0.3 & 0.2\end{bmatrix}=N_3$$
$$N_2=\begin{bmatrix}0.9 & 0.1\\ -0.1 & 0.5\end{bmatrix}=N_4,\ d_m=0.9,\ d_M=1.1 \tag{40}$$

In the remaining of this section the following parameters are also needed:

$$\alpha=0.9, \beta=0.8, g=0.8, \gamma=100, \xi=0.95, V_1=\begin{bmatrix}1.39 & 1.02\\ 0.23 & 0.36\end{bmatrix}, V_2=\begin{bmatrix}-1.39 & 1.02\\ 0.23 & -0.36\end{bmatrix}$$

We opt for $G=\begin{bmatrix}2.19 & 2.19\end{bmatrix}$ which guarantees the fact that (GB) is invertible. The initial state and estimated state conditions are respectively selected to be $x(0)=\begin{bmatrix}0.4 & 10.9716\end{bmatrix}^T$, $\hat{x}(0)=\begin{bmatrix}0.4 & 0.6716\end{bmatrix}^T$, while the initial input is fixed to $u(0)=0$
Based on the Theorem 2 and by means of the CVX MATLAB toolbox, we compute respectively the observer and the controller gain matrices $L=\begin{bmatrix}-0.9712 & 0.919\end{bmatrix}^T$ and $K=\begin{bmatrix}0.0519 & -0.71\end{bmatrix}^T$.
Since the system (1) is affected by nonlinearities $f(x(k), x(k-d(k)))$ of the form (3), then in the current example, we assume that the i-th component $i=1,2$ of such function are defined as follows:

$$f_i(x_i(k), x_i(k-d(k)))=\begin{cases}\cos\left(\dfrac{2x_i(k)x_i^2(k-d(k))}{x_i^2(k)+x_i^2(k-d(k))}\right) & \text{if } \begin{matrix}x_i(k)\neq 0 \text{ and}\\ x_i(k-d(k))\neq 0\end{matrix}\\ 0 \quad \text{elsewhere} & \end{cases} \tag{41}$$

This nonlinearity, when it is in function of the state vector, i.e (denoted by $f(x(k), x(k-d(k)))$), it is then depicted in Fig. 1. However, when such nonlinearity is expressed in function of the estimated state vector, i.e designed by $f(\hat{x}(k), \hat{x}(k-d(k)))$, it is thus plotted in Fig. 2.

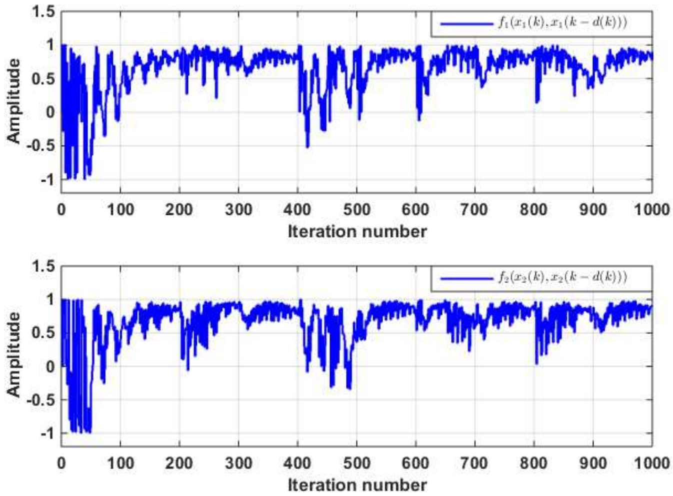

Fig. 1. Both components of $f(x(k), x(k-d(k)))$.

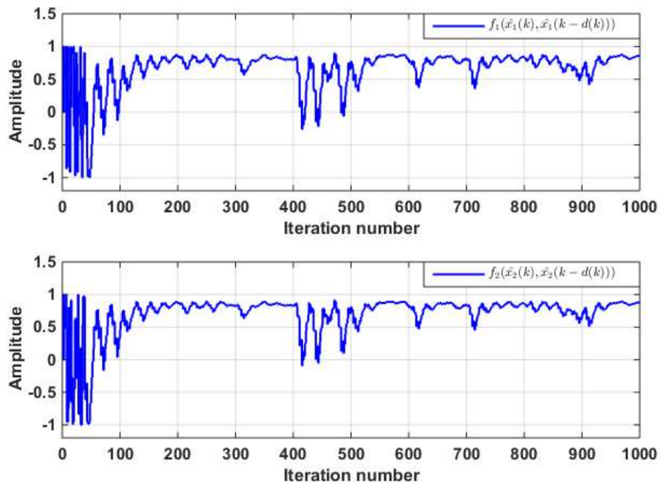

Fig. 2. Both components of $f(\hat{x}(k), \hat{x}(k-d(k)))$.

To show how the system (40) behaves under the presence of disturbances and when the proposed controller is applied, and in order to judge the validity of the proposed control approach. We have injected different types of disturbance signals: In fact, we have considered two different waveforms: a sawtooth and a square signals with random amplitudes. Both components of the external disturbance w_1 and w_2 are illustrated in Fig. 3.

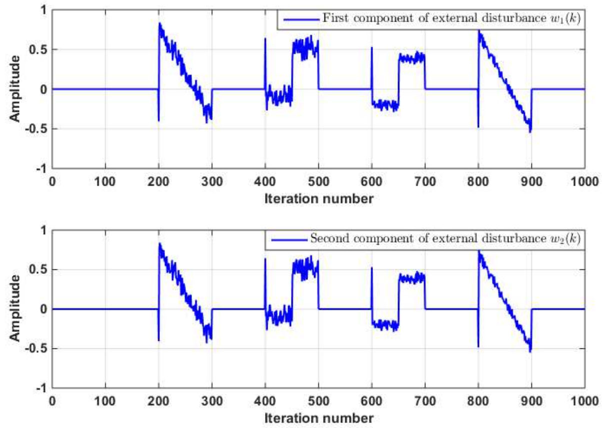

Fig. 3. Disturbance signal $w(k)$.

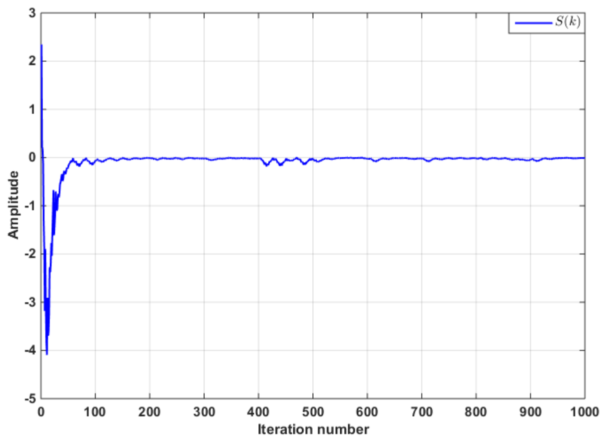

Fig. 4. Sliding surface $S(k)$.

The sliding surface and the control input are respectively depicted in Fig. 4 and Fig. 5. The state x_1 and its estimates $\hat{x}_1$ are shown in Fig. 6, while the state x_2 and its estimates $\hat{x}_2$ are presented in Fig. 7. The output y and the estimated output $\hat{y}$ are reported in Fig. 8. From these reported results, it have been confirmed that our proposed

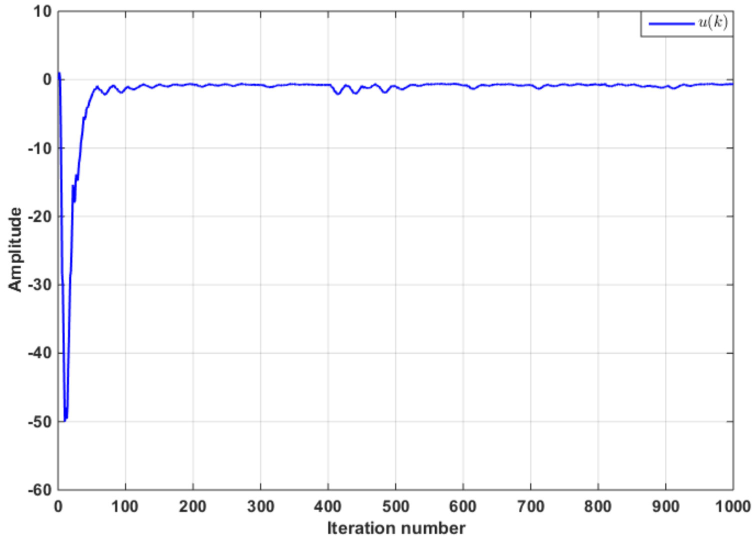

Fig. 5. Control input $u(k)$.

Observer-based discrete sliding mode control allows the convergence of the system estimated states toward the system real states. Moreover, the designed control approach provides finite-time convergence to a band around zero inspite of the presence of time-varying delay, UBB uncertainties, external disturbances and nonlinearities as well.

5 Conclusion

In this paper, the design of a new robust observer has been investigated for uncertain discrete time-delay systems, by using the sliding mode control technique. The studied system is affected by time-varying delay on the state, unmatched UBB uncertainties, external disturbances and Hölder nonlinearities as well. A new less conservative delay-dependent sufficient condition guaranteeing the robust asymptotic stability of the closed-loop system is derived in terms of LMIs. Based on that condition, a novel sliding surface based observer has been designed and a new discrete-time sliding mode control law is developed. Simulation results have asserted the effectiveness of the proposed control scheme.

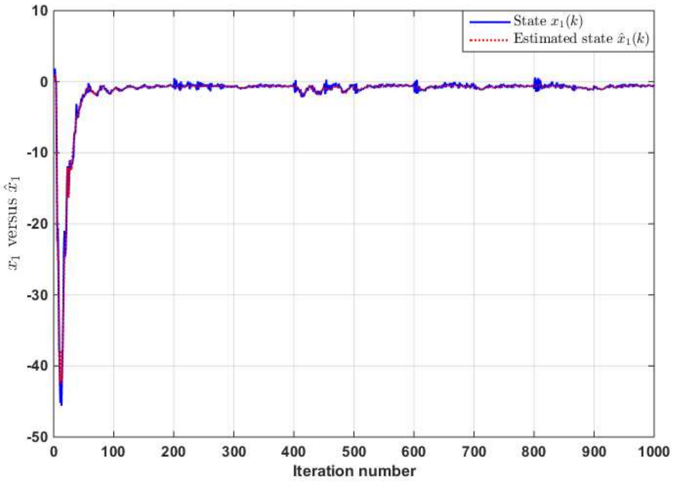

Fig. 6. The state $x_1(k)$ versus the estimated state $\hat{x}_1(k)$.

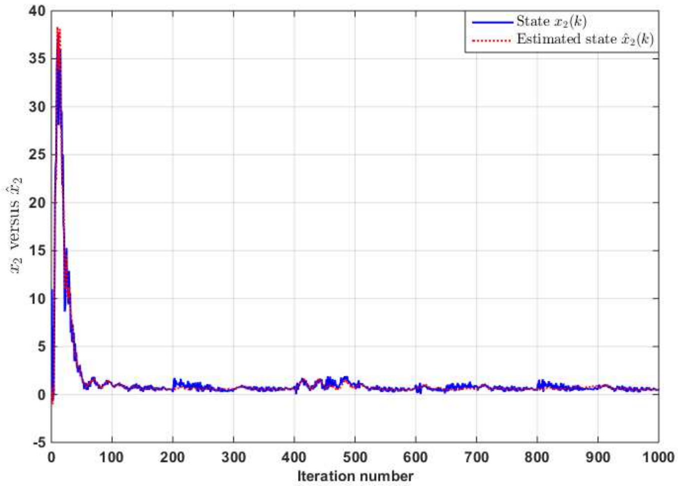

Fig. 7. The state $x_2(k)$ versus the estimated state $\hat{x}_2(k)$.

References

1. El Ghaoui, L., Niculescu, S.:Advances in linear matrix inequality methods in control. Society for Industrial and Applied Mathematics, Philadelphia PA (2000)
2. Fridman, I.: Introduction to time-delay systems: Analysis and control. Springer, New York (2014)

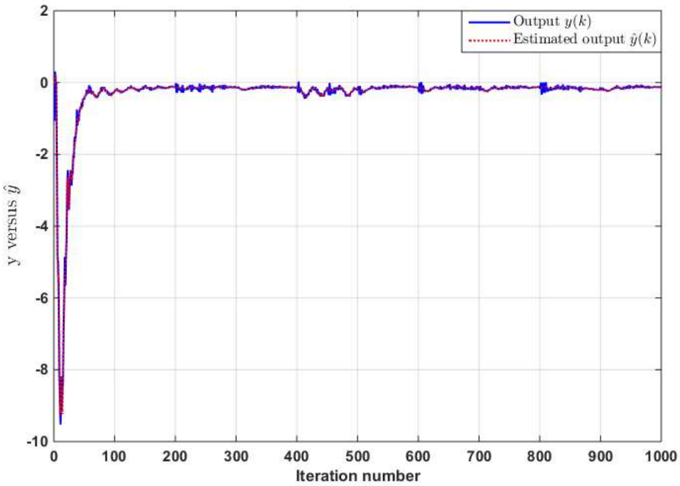

Fig. 8. The output $y(k)$ versus the estimated output $\hat{y}(k)$.

3. Ghrab, S., Benamor, A., Messaoud, H.: Robust discrete-time sliding mode control for systems with time-varying state delay and uncertainties on state and control input. Transactions of the Institute of Measurement and Control 39:9, 1293–1312 (2017). doi:10.1177/0142331216636954
4. Han, Y., Kao, Y., Gao, C.: Robust observer-based H_∞ control for uncertain discrete singular systems with time-varying delays via sliding mode approach. ISA transactions 81, 81–88 (2018). doi:10.1016/j.isatra.2018.05.023
5. Li, S., Yang, J., Chen, W., Chen, X.: Disturbance observer-based control: methods and applications. CRC press, USA (2016)
6. Muñoz-Vázquez, AJ., Parra-Vega, V., Sánchez-Orta, A., Romero-Galván, G.: Finite-time disturbance observer via continuous fractional sliding modes. Transactions of the Institute of Measurement and Control 40:14, 3953–3963 (2018). doi:10.1177/0142331217737833
7. Sayyaddelshad, S., Gustafsson, T.: H∞ observer design for uncertain nonlinear discrete-time systems with time-delay: LMI optimization approach. International Journal of Robust and Nonlinear Control 147, 195-197 (1981). doi:10.1002/rnc.3155
8. Spurgeon, S.: Sliding mode observers: a survey. International Journal of Systems Science 39: 8 751–764 (2008). doi:10.1080/00207720701847638
9. Sun, N., Zhang, Y., Songi, G., Li, T.: Design of observer-based quantized control for nonlinear time-delay systems. Transactions of the Institute of Measurement and Control 40-15, 4220-4232 (2018). doi:10.1177/0142331217745581
10. Richard, JP.: Time-delay systems: an overview of some recent advances and open problems. Automatica 39:10, 1667–1694 (2003). doi:10.1016/S0005-1098(03)00167-5
11. Tan, CP., Edwards, C.: An LMI approach for designing sliding mode observers. International Journal of Control 74:16, 1559–1568, (2001). doi:10.10.1109/CDC.2000.914193
12. Wang, X., Zuo, X., Liu, J., Liang, H. : Robust observer-based control for uncertain discrete time-delay systems with nonlinearities electric-hydraulic system under Hölder condition. Optimal Control Applications and Methods 38: 6, 1120–1131 (2017). doi:10.1002/oca.2318