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► **To cite this version:**

Martin Bunder, Abderrahmane Nitaj, Willy Susilo, Joseph Tonien. Cryptanalysis of RSA-type cryptosystems based on Lucas sequences, Gaussian integers and elliptic curves. *Journal of information security and applications*, 2018, 10.1016/j.jisa.2018.04.006 . hal-02320970

**HAL Id: hal-02320970**

**<https://hal-normandie-univ.archives-ouvertes.fr/hal-02320970>**

Submitted on 20 Oct 2019

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# Cryptanalysis of RSA-type cryptosystems based on Lucas sequences, Gaussian integers and elliptic curves<sup>\*</sup>

Martin Bunder<sup>1</sup>, Abderrahmane Nitaj<sup>2</sup>, Willy Susilo<sup>3</sup>, and Joseph Tonien<sup>3</sup>

<sup>1</sup> School of Mathematics and Applied Statistics,  
University of Wollongong, Australia  
mbunder@uow.edu.au

<sup>2</sup> Département de Mathématiques,  
Université de Caen, France  
abderrahmane.nitaj@unicaen.fr

<sup>3</sup> Centre for Computer and Information Security Research,  
School of Computing and Information Technology,  
University of Wollongong, Australia  
[wsusilo,joseph\_tonien]@uow.edu.au

**Abstract.** In 1995, Kuwakado, Koyama and Tsuruoka presented a new RSA-type scheme based on singular cubic curves  $y^2 \equiv x^3 + bx^2 \pmod{N}$  where  $N = pq$  is an RSA modulus. Then, in 2002, Elkamchouchi, Elshenawy and Shaban introduced an extension of the RSA scheme to the field of Gaussian integers using a modulus  $N = PQ$  where  $P$  and  $Q$  are Gaussian primes such that  $p = |P|$  and  $q = |Q|$  are ordinary primes. Later, in 2007, Castagnos proposed a scheme over quadratic field quotients with an RSA modulus  $N = pq$  based on Lucas sequences. In the three schemes, the public exponent  $e$  is an integer satisfying the key equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$ . In this paper, we apply the continued fraction method to launch an attack on the three schemes when the private exponent  $d$  is sufficiently small. Our experiments demonstrate that for a 1024-bit modulus, our method works for values of  $d$  of up to 520 bits. We also examine the effect of dropping the usual assumption that  $p$  and  $q$  have the same bit size.

**Keywords:** RSA, elliptic curves, continued fractions.

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<sup>\*</sup> This paper is the extended version of “A new attack on three variants of the RSA cryptosystem” [2] presented at ACISP 2016

## 1 Introduction

The public key cryptosystem RSA was introduced by Rivest, Shamir and Adleman [11] in 1978. It is the most popular and widely used public-key cryptosystem. The RSA operations system are based on modular arithmetic. Let  $p$  and  $q$  be two large primes. The product  $N = pq$  is called the RSA modulus and the product  $\phi(N) = (p - 1)(q - 1)$  is the Euler totient function. In RSA, the public exponent  $e$  and the private exponent  $d$  are integers satisfying  $ed \equiv 1 \pmod{\phi(N)}$ . A message  $m$  is encrypted as  $c \equiv m^e \pmod{N}$  and decrypted using  $m \equiv c^d \pmod{N}$ .

Since its introduction, the RSA cryptosystem has been generalized in various ways, including extensions to singular elliptic curves and Gaussian integers.

In 1995, Kuwakado, Koyama and Tsuruoka [9] presented a new RSA-type scheme based on singular cubic curves with equation  $y^2 \equiv x^3 + bx^2 \pmod{N}$  where  $N = pq$  is an RSA modulus and  $b \in \mathbb{Z}/N\mathbb{Z}$ . The public exponent is an integer  $e$  such that  $\gcd(e, (p^2 - 1)(q^2 - 1)) = 1$  and the decryption exponent is the integer  $d \equiv e^{-1} \pmod{(p^2 - 1)(q^2 - 1)}$ . From this, we deduce that  $e$  and  $d$  satisfy a key equation of the form  $ed - k(p^2 - 1)(q^2 - 1) = 1$  where  $k$  is a positive integer.

In 2002, Elkamchouchi, Elshenawy and Shaban [6] introduced an extension of RSA to the ring of Gaussian integers. A Gaussian integer is a complex number of the form  $a + ib$  where both  $a$  and  $b$  are integers and  $i^2 = -1$ . The set of all Gaussian integers is denoted  $\mathbb{Z}[i]$ . A Gaussian prime number is a Gaussian integer that cannot be represented as a product of non-unit Gaussian integers. The only unit Gaussian integers are  $\pm 1, \pm i$ . Let  $P = a + ib$  and  $Q = a' + ib'$  be two Gaussian primes. Consider the Gaussian integer  $N = PQ$  and the Euler totient function  $\phi(N) = (|P| - 1)(|Q| - 1) = (a^2 + b^2 - 1)(a'^2 + b'^2 - 1)$ . Let  $e$  be an integer such that  $d \equiv e^{-1} \pmod{\phi(N)}$  exists. Then, in the RSA scheme over the domain of Gaussian integers, a message  $m \in \mathbb{Z}[i]$  is encrypted using  $c \equiv m^e \pmod{N}$  and decrypted using  $m \equiv c^d \pmod{N}$ . We note that, in this RSA variant, the key equation is  $ed - k(|P| - 1)(|Q| - 1) = 1$  for  $N = PQ \in \mathbb{Z}[i]$ . In the situation that  $N = pq$  is an ordinary RSA modulus, the key equation becomes  $ed - k(p^2 - 1)(q^2 - 1) = 1$ , which is the same than in the Kuwakado-Koyama-Tsuruoka elliptic curve variant of RSA.

In 2007, Castagnos [4] proposed a probabilistic scheme based on an RSA modulus  $N = pq$  and using arithmetical operations in quadratic field quotients. Let  $e$  be an integer such that  $\gcd(e, (p^2 - 1)(q^2 - 1)) = 1$ . For any integer  $r$ , let  $V_e(r)$  be the  $e$ th term of the Lucas sequence defined by  $V_0(r) = 2, V_1(r) = r$  and  $V_{k+2} = rV_{k+1}(r) - V_k(r)$  for  $k \geq 0$ . In this scheme, a message  $m \in \mathbb{Z}/N\mathbb{Z}$  is encrypted using  $c \equiv (1 + mN)V_e(r) \pmod{N^2}$  where  $r$  is a random integer with  $2 \leq r \leq N - 2$ . Then some arithmetical properties, one can decrypt  $c$  to get the original message  $m$ . Similarly to the Kuwakado-Koyama-Tsuruoka elliptic curve variant of RSA and RSA with Gaussian integers, Castagnos scheme leads to the key equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$ .

The security of the RSA cryptosystem and its variants are based on the difficulty of factoring large integers of the shape  $N = pq$ . Nevertheless, in some

cases, the modulus  $N$  can be factored by algebraic methods that are not based on factoring algorithms. For example, in 1990, Wiener [12] showed how to break the RSA when the decryption exponent  $d$  satisfies  $d < \frac{1}{3}N^{0.25}$ . Wiener's method is based on solving the key equation  $ed - k(p-1)(q-1) = 1$  by applying the continued fraction algorithm to the public rational fraction  $\frac{e}{N}$ . When  $d$  is small enough,  $\frac{k}{d}$  is one of the convergents of the continued fraction expansion of  $\frac{e}{N}$ . Later, Boneh and Durfee [1] applied lattice reduction and Coppersmith's technique [5] and extended the bound to  $d < N^{0.292}$ . Recently, using the convergents of the continued fraction expansion of  $\frac{e}{N'}$  where  $N'$  is a number depending on  $N$ , Bunder and Tonien [3] could break the RSA if  $d^2e < 8N^{1.5}$ .

The complexity of the encryption and decryption algorithms are based on the size of the encryption key  $e$  and the size of decryption key  $d$ , respectively. In a cryptosystem with a limited resource such as a credit card, it is desirable to have a smaller value of  $d$ . In some scenario, for convenience,  $e$  is set to a small constant, such as  $e = 3$ .

In this paper, we consider one of the following scenarios where  $N = pq$  is the product of two large primes and the public exponent  $e$  satisfies an equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$  with a suitably small secret exponent  $d$ :

- an instance of the Kuwakado-Koyama-Tsuruoka cryptosystem [9],
- an instance of the RSA over Gaussian integers [6],
- an instance of Castagnos scheme [4].

Our method is inspired by Bunder and Tonien's technique [3]. We show that in the case when  $p$  and  $q$  has equal bit length, if  $d^2e < 2N^3 - 18N^2$  then one can find  $p$  and  $q$  and then factor the modulus  $N$ . Our method is based on the continued fraction algorithm as in Bunder and Tonien's attack. Under the condition  $d < \sqrt{\frac{2N^3 - 18N^2}{e}}$ , we show that one can find  $\frac{k}{d}$  among the convergents of the continued fraction expansion of the public rational number  $\frac{e}{N^2 - \frac{9}{4}N + 1}$ . In the general case, where  $q < p < \mu q$ , if  $d^2e < \frac{\mu N^3}{(\mu-1)^2}$  then we can find  $\frac{k}{d}$  among the convergents of the continued fraction expansion of  $\frac{e}{N^2 + 1 - \frac{(\mu+1)^2}{2\mu}N}$ .

The paper is organized as follows. In Section 2, we present the Kuwakado-Koyama-Tsuruoka RSA-type scheme, the RSA scheme over Gaussian integers and the Castagnos scheme. In Section 3, we review some facts and lemmas used in our attack. In Section 4, we present our new attack for the case  $q < p < 2q$  and in Section 5, our attack for the general case  $q < p < \mu q$ . Experiment results for  $\mu = 2$  and  $\mu = 6$  are presented in Section 6. Note that the contents of Section 5 and Section 6 are new and have not been included in our conference paper [2].

## 2 Preliminaries

In this section, we present the three variants of the RSA cryptosystem for which our attack works, namely the Kuwakado-Koyama-Tsuruoka RSA-type scheme, the RSA scheme over Gaussian integers and the Castagnos scheme.

## 2.1 The Kuwakado-Koyama-Tsuruoka RSA-type scheme

The Kuwakado-Koyama-Tsuruoka RSA-type scheme is based on the use of an RSA modulus  $N = pq$  as the modulus of a singular elliptic curve. Let  $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$  be the ring of integers modulo  $N$  and  $\mathbb{F}_p$  be the finite field. Let  $a$  and  $b$  be integers with  $\gcd(ab, N) = 1$  and  $\gcd(4a^3 + 27b^2, N) = 1$ . A singular elliptic curve  $E_N(a, b)$  over the ring  $\mathbb{Z}_N$  is the concatenation of a point  $\mathcal{O}_N$ , called the point at infinity, and the set of points  $(x, y) \in \mathbb{Z}_N^2$  satisfying the Weierstrass equation

$$y^2 + axy \equiv x^3 + bx^2 \pmod{N}.$$

If we consider this form modulo  $p$ , we get an elliptic curve  $E_p(a, b)$  over  $\mathbb{F}_p$

$$E_p(a, b) : y^2 + axy \equiv x^3 + bx^2 \pmod{p},$$

with the point at infinity  $\mathcal{O}_p$ . It is well known that the chord-and-tangent method defines an addition law on singular elliptic curves, as for all elliptic curves on  $\mathbb{F}_p$ . The addition law can be summarized as follows.

- For any point  $P \in E_p(a, b)$ ,  $P + \mathcal{O}_p = \mathcal{O}_p + P = P$ .
- If  $P = (x, y) \in E_p(a, b)$ , then  $-P = (x, -ax - y)$ .
- If  $P = (x, y)$ , then  $2P = P_3 = (x_3, y_3)$  with

$$x_3 = \left( \frac{3x^2 + 2bx - ay}{2ay + ax} \right)^2 + a \left( \frac{3x^2 + 2bx - ay}{2ay + ax} \right) - b - 2x,$$

$$y_3 = - \left( \frac{3x^2 + 2bx - ay}{2ay + ax} + a \right) x_3 - \frac{-x^3}{2ay + ax}.$$

- If  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$  with  $P_1 \neq \pm P_2$ , then  $P_1 + P_2 = P_3 = (x_3, y_3)$  with

$$x_3 = \left( \frac{y_2 - y_1}{x_2 - x_1} \right)^2 + a \left( \frac{y_2 - y_1}{x_2 - x_1} \right) - b - x_1 - x_2,$$

$$y_3 = - \left( \frac{y_2 - y_1}{x_2 - x_1} + a \right) x_3 - \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}.$$

The addition law can be extended to the elliptic curve  $E_N(a, b)$  in the same way as the addition in  $E_p(a, b)$  by replacing computations modulo  $p$  by computations modulo  $N$ . In  $E_N(a, b)$ , a specific problem can occur. Sometimes, the inverse modulo  $N$  does not exist. In this case, this could lead to finding a prime factor of  $N$ , which is unlikely to happen when  $p$  and  $q$  are large. Note that this is one of the principles of Elliptic Curve Method of factorization [10].

In 1995, Kuwakado, Koyama and Tsuruoka [9] proposed a system based on singular elliptic curves modulo an RSA modulus, which can be summarized as follows.

**1. Key Generation:**

- Choose two distinct prime numbers  $p$  and  $q$  of similar bit-length.
- Compute  $N = pq$ .
- Choose  $e$  such that  $\gcd(e, (p^2 - 1)(q^2 - 1)) = 1$ .
- Compute  $d = e^{-1} \pmod{(p^2 - 1)(q^2 - 1)}$ .
- Keep  $p, q, d$  secret and publish  $N, e$ .

**2. Encryption:**

- Transform the message as  $m = (m_x, m_y) \in \mathbb{Z}_N \times \mathbb{Z}_N$ .
- Compute  $b = \frac{m_y^2 - m_x^3}{m_x^2} \pmod{N}$ .
- Compute the ciphertext point  $(c_x, c_y) = e(m_x, m_y)$  on the elliptic curve  $y^2 = x^3 + bx^2 \pmod{N}$ .

**3. Decryption:**

- Compute  $b = \frac{c_y^2 - c_x^3}{c_x^2} \pmod{N}$ .
- Compute the plaintext point  $(m_x, m_y) = d(c_x, c_y)$  on the elliptic curve  $y^2 = x^3 + bx^2 \pmod{N}$ .

Observe the modular inverse  $d = e^{-1} \pmod{(p^2 - 1)(q^2 - 1)}$  can be transformed as a key equation

$$ed - k(p^2 - 1)(q^2 - 1) = 1,$$

which will be the starting equation of our new attack.

**2.2 RSA over the domain of Gaussian integers**

We now focus on how to extend the RSA cryptosystem to the ring of Gaussian integers. We begin by reviewing the main properties of Gaussian integers.

A Gaussian integer is a complex number of the form  $a + bi$  where  $a, b \in \mathbb{Z}$  and  $i^2 = -1$ . The set of all Gaussian integers is the ring  $\mathbb{Z}[i]$ . Let  $\alpha$  and  $\beta \neq 0$  be two Gaussian integers. We say that  $\beta$  divides  $\alpha$  if there exists a Gaussian integer  $\gamma$  such that  $\alpha = \beta\gamma$ . The norm of a Gaussian integer  $a + bi$  is  $|a + bi| = a^2 + b^2$ . A Gaussian prime is a Gaussian integer which is divisible only by a unit. The units in  $\mathbb{Z}[i]$  are  $\pm 1$  and  $\pm i$  and have norm 1. As a consequence, if  $a^2 + b^2$  is a prime number in  $\mathbb{Z}$ , then  $a + ib$  is a Gaussian prime. Conversely, if  $p \in \mathbb{Z}$  is an ordinary prime number, then Gaussian integers  $p$  and  $pi$  are Gaussian primes if and only if  $p \equiv 3 \pmod{4}$ . The existence of prime factorization in  $\mathbb{Z}[i]$  allows us to consider Gaussian integers of the form  $N = PQ$  where  $P$  and  $Q$  are Gaussian primes with large norm. Similarly, the existence of Euclidean division and Euclidean algorithm in  $\mathbb{Z}[i]$  allow us to consider arithmetic operations modulo  $N$ . On the other hand, if  $P$  is a Gaussian prime, then  $\alpha^{|P|-1} \equiv 1 \pmod{P}$  whenever  $\alpha \not\equiv 0 \pmod{P}$ . Similarly, if  $N = PQ$  is the product of two Gaussian primes, then  $\alpha^{(|P|-1)(|Q|-1)} \equiv 1 \pmod{N}$  whenever  $\alpha \not\equiv 0 \pmod{N}$ . In particular, if  $N = pq \in \mathbb{Z}$  is the product of two ordinary primes, then  $\alpha^{(p^2-1)(q^2-1)} \equiv 1 \pmod{N}$  whenever  $\alpha \not\equiv 0 \pmod{N}$ .

Using the arithmetical operations on the ring  $\mathbb{Z}[i]$ , Elkamchouchi, Elshenawy and Shaban [6] proposed an extension of the RSA cryptosystem to Gaussian integers. The scheme can be summarized as follows.

**1. Key Generation:**

- Choose two distinct Gaussian primes  $P$  and  $Q$  of similar norm.
- Compute  $N = PQ$ .
- Choose  $e$  such that  $\gcd(e, (|P| - 1)(|Q| - 1)) = 1$ .
- Determine  $d = e^{-1} \pmod{(|P| - 1)(|Q| - 1)}$ .
- Keep  $P, Q, d$  secret, publish  $N, e$ .

**2. Encryption:**

- Transform the message as a Gaussian integer  $M \in \mathbb{Z}[i]$ .
- Compute  $C \equiv M^e \pmod{N}$ .

**3. Decryption:**

- Compute  $M \equiv C^d \pmod{N}$ .

When  $N = pq \in \mathbb{Z}$  where  $p$  and  $q$  are ordinary prime numbers of the form  $4m+3$ , the modular inverse of  $e$  becomes  $d = e^{-1} \pmod{(p^2 - 1)(q^2 - 1)}$  and can be rewritten as

$$ed - k(p^2 - 1)(q^2 - 1) = 1.$$

This is the same key equation that comes up in the Kuwakado-Koyama-Tsuruoka RSA-type scheme.

### 2.3 Castagnos scheme

Castagnos scheme [4] was proposed in 2007 and uses an RSA modulus  $N = pq$  and a public exponent  $e$  such that  $\gcd(e, (p^2 - 1)(q^2 - 1)) = 1$ . The encryption and the decryption algorithms make use of the Lucas series. Let  $r$  be an integer. Define  $V_0(r) = 2$  and  $V_1(r) = r$ . For  $k \geq 0$ , the  $k + 2$ th term of the Lucas sequence is defined by  $V_{k+2} = rV_{k+1}(r) - V_k(r)$ . The Lucas series can be computed efficiently by the square and multiply algorithm. The Castagnos scheme can be summarized as follows, where  $\left(\frac{x}{p}\right)$  is the Jacobi symbol.

**1. Key Generation:**

- Choose two distinct prime numbers  $p$  and  $q$  of similar bit-length.
- Compute  $N = pq$ .
- Choose  $e$  such that  $\gcd(e, (p^2 - 1)(q^2 - 1)) = 1$ .
- Keep  $p, q$  secret and publish  $N, e$ .

**2. Encryption:**

- Transform the message as an integer  $m \in \mathbb{Z}/N\mathbb{Z}$ .
- Choose a random integer  $r \in [2, n - 2]$ .
- Compute the ciphertext  $c \equiv (1 + mN)V_e(r) \pmod{N^2}$ .

### 3. Decryption:

- Compute  $i_p \equiv \left(\frac{c^2-4}{p}\right)$  and  $d(p, i_p) \equiv e^{-1} \pmod{p - i_p}$ .
- Compute  $i_q \equiv \left(\frac{c^2-4}{q}\right)$  and  $d(q, i_q) \equiv e^{-1} \pmod{q - i_q}$ .
- Compute  $r_p \equiv V_{d(p, i_p)} \pmod{p}$  and  $r_q \equiv V_{d(q, i_q)} \pmod{q}$ .
- Compute  $p' \equiv p^{-1} \pmod{q}$  and  $r = r_p + p(r_p - r_q)p' \pmod{N}$ .
- Compute  $t_p \equiv \frac{c}{V_e(r)} \pmod{p^2}$  and  $m_p \equiv \frac{t_p-1}{p} \cdot q^{-1} \pmod{p}$ .
- Compute  $t_q \equiv \frac{c}{V_e(r)} \pmod{q^2}$  and  $m_q \equiv \frac{t_q-1}{q} \cdot p^{-1} \pmod{q}$ .
- Compute the plaintext  $m \equiv m_p + p(m_q - m_p)p' \pmod{N}$ .

Despite the inverse  $d \equiv e^{-1} \pmod{(p^2 - 1)(q^2 - 1)}$  is not being used directly in the scheme, we use the key equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$  to launch an attack on Castagnos scheme when  $d$  is suitably small.

## 3 Useful lemmas

In this section, we review the main properties of the continued fractions and state a useful lemma that will be used in the attack.

A **continued fraction** is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

The continued fraction expansion of a number is formed by subtracting away the integer part of it and inverting the remainder and then repeating this process again and again. The coefficients  $a_i$  of the continued fraction of a number  $x$  are constructed as follows:

$$x_0 = x, \quad a_n = [x_n], \quad x_{n+1} = \frac{1}{x_n - a_n}$$

We use the following notation to denote the continued fraction

$$x = [a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}}$$

If  $k \leq n$ , the continued fraction  $[a_0, a_1, \dots, a_k]$  is called the  $k^{\text{th}}$  convergent of  $x$ . The following theorem gives us the *fundamental recursive formulas* to calculate the convergents.



**Theorem 1.** [7] The  $k^{\text{th}}$  convergent can be determined as

$$[a_0, \dots, a_k] = \frac{p_k}{q_k}$$

where the sequences  $\{p_n\}$  and  $\{q_n\}$  are specified as follows:

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_n &= a_n p_{n-1} + p_{n-2}, & \forall n \geq 0, \\ q_{-2} &= 1, & q_{-1} &= 0, & q_n &= a_n q_{n-1} + q_{n-2}, & \forall n \geq 0. \end{aligned}$$

**Theorem 2.** [7] Let  $p, q$  be positive integers such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

then  $\frac{p}{q}$  is a convergent of the continued fraction of  $x$ .

Now, we present a useful result that will be used throughout the paper.

**Lemma 1.** Let  $N = pq$  be an RSA modulus with  $q < p < 2q$ . Let  $\phi_1 = N^2 + 1 - \frac{5}{2}N$  and  $\phi_2 = N^2 + 1 - 2N$ . Then

$$\phi_1 < (p^2 - 1)(q^2 - 1) < \phi_2.$$

*Proof.* Suppose that  $q < p < 2q$ . Then  $1 < \frac{p}{q} < 2$ , so since the function  $f(x) = x + \frac{1}{x}$  is increasing on  $[1, +\infty)$ , we get  $f(1) < f\left(\frac{p}{q}\right) < f(2)$ , that is

$$2 < \frac{p}{q} + \frac{q}{p} < \frac{5}{2}.$$

Multiplying by  $N$ , we get

$$2N < p^2 + q^2 < \frac{5}{2}N.$$

Since  $(p^2 - 1)(q^2 - 1) = N^2 + 1 - (p^2 + q^2)$ , we get

$$N^2 + 1 - \frac{5}{2}N < (p^2 - 1)(q^2 - 1) < N^2 + 1 - 2N,$$

that is  $\phi_1 < (p^2 - 1)(q^2 - 1) < \phi_2$ . This terminates the proof.

## 4 A new attack on RSA variants based on continued fractions

In this section, we propose a new attack on the Kuwakado-Koyama-Tsuruoka cryptosystem as well as RSA over the Gaussian integer domain and the Castagnos scheme in the situation that the key equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$  is satisfied with a suitably small secret exponent  $d$ .

**Theorem 3.** *Let  $(N, e)$  be a public key in the Kuwakado-Koyama-Tsuruoka cryptosystem or in the RSA cryptosystem with Gaussian integers or in the Castagnos scheme with  $N = pq$  and  $q < p < 2q$ . If  $e < (p^2 - 1)(q^2 - 1)$  satisfies an equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$  with*

$$d < \sqrt{\frac{2N^3 - 18N^2}{e}},$$

then  $\frac{k}{d}$  is a convergent of the continued fraction expansion of

$$\frac{e}{N^2 - \frac{9}{4}N + 1}$$

and one can factor  $N$  in polynomial time.

*Proof.* Let  $\phi_1 = N^2 + 1 - \frac{5}{2}N$  and  $\phi_2 = N^2 + 1 - 2N$ . Then  $N' = N^2 - \frac{9}{4}N + 1$  is the midpoint of the interval  $[\phi_1, \phi_2]$ . Since  $(p^2 - 1)(q^2 - 1) \in [\phi_1, \phi_2]$ , then

$$|(p^2 - 1)(q^2 - 1) - N'| < \frac{1}{2}(\phi_2 - \phi_1) = \frac{1}{4}N. \quad (1)$$

Using the equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$ , we get

$$\begin{aligned} \left| \frac{e}{N'} - \frac{k}{d} \right| &\leq e \left| \frac{1}{N'} - \frac{1}{(p^2 - 1)(q^2 - 1)} \right| + \left| \frac{e}{(p^2 - 1)(q^2 - 1)} - \frac{k}{d} \right| \\ &= e \frac{|(p^2 - 1)(q^2 - 1) - N'|}{N'(p^2 - 1)(q^2 - 1)} + \frac{1}{(p^2 - 1)(q^2 - 1)d} \end{aligned}$$

Then, using  $d = \frac{k(p^2 - 1)(q^2 - 1) + 1}{e}$  and (3), we get

$$\left| \frac{e}{N'} - \frac{k}{d} \right| < \frac{eN}{4N'(p^2 - 1)(q^2 - 1)} + \frac{e}{(p^2 - 1)(q^2 - 1)(k(p^2 - 1)(q^2 - 1) + 1)}.$$

Now, using Lemma 1, we get

$$\left| \frac{e}{N'} - \frac{k}{d} \right| < \frac{eN}{4\phi_1^2} + \frac{e}{\phi_1^2} < \frac{e(N + 4)}{4(\phi_1 - 1)^2} = \frac{e(N + 4)}{4(N^2 - \frac{5}{2}N)^2}. \quad (2)$$

A straightforward calculation shows that

$$\frac{N + 4}{4(N^2 - \frac{5}{2}N)^2} < \frac{1}{4N^3 - 36N^2}.$$

Combining this with (2), we get

$$\left| \frac{e}{N'} - \frac{k}{d} \right| < \frac{e}{4N^3 - 36N^2}.$$

If  $d < \sqrt{\frac{2N^3 - 18N^2}{e}}$ , then  $\left| \frac{e}{N'} - \frac{k}{d} \right| < \frac{1}{2d^2}$  and by Theorem 2,  $\frac{k}{d}$  is a convergent of the continued fraction expansion of  $\frac{e}{N'}$ . Using  $k$  and  $d$ , we get

$$(p^2 - 1)(q^2 - 1) = \frac{ed - 1}{k}.$$

Combining with  $N = pq$ , we get the values of  $p$  and  $q$  which leads to the factorization of  $N$ . Observe that every step in the proof can be done in polynomial time. This terminates the proof.

## 5 Different bit size primes

In Section 4, we consider the case where the two primes  $p$  and  $q$  are of the same bit size, i.e.  $q < p < 2q$ . In this section, we give a generalised attacked by considering two primes  $p$  and  $q$  of arbitrary sizes. We will not require  $p$  and  $p$  to have the same bit size, but instead, we consider the case  $q < p < \mu q$  where  $\mu$  is a parameter. We show that if

$$d^2 e < \frac{\mu N^3}{(\mu - 1)^2}$$

then the three RSA-variant schemes can be broken.

**Theorem 4.** *Let  $(N, e)$  be a public key in the Kuwakado-Koyama-Tsuruoka cryptosystem or in the RSA cryptosystem with Gaussian integers or in the Castagnos scheme with  $N = pq$  and  $q < p < \mu q$ . If  $e < (p^2 - 1)(q^2 - 1)$  satisfies an equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$  with*

$$d < \frac{N(N - (\mu + \frac{1}{\mu}))}{\sqrt{e(\frac{(\mu-1)^2}{\mu}N + 2)}} \approx \frac{\sqrt{\mu} N^{\frac{3}{2}}}{(\mu - 1) \sqrt{e}}$$

then  $\frac{k}{d}$  is a convergent of the continued fraction expansion of

$$\frac{e}{N^2 + 1 - \frac{(\mu+1)^2}{2\mu}N}$$

and one can factor  $N$  in polynomial time.

*Proof.* We have  $1 < \frac{p}{q} < \mu$ , so based on the increasing property of the function  $f(x) = x + \frac{1}{x}$  on  $[1, +\infty)$ , we get  $f(1) < f\left(\frac{p}{q}\right) < f(\mu)$ , that is

$$2 < \frac{p}{q} + \frac{q}{p} < \mu + \frac{1}{\mu}.$$

Multiplying by  $N$ , we get

$$2N < p^2 + q^2 < (\mu + \frac{1}{\mu})N.$$

Since  $(p^2 - 1)(q^2 - 1) = N^2 + 1 - (p^2 + q^2)$ , we get

$$N^2 + 1 - (\mu + \frac{1}{\mu})N < (p^2 - 1)(q^2 - 1) < N^2 + 1 - 2N.$$

Let

$$\begin{aligned}\phi_1 &= N^2 + 1 - (\mu + \frac{1}{\mu})N, \\ \phi_2 &= N^2 + 1 - 2N, \\ N' &= N^2 + 1 - \frac{(\mu + 1)^2}{2\mu}N.\end{aligned}$$

Then  $N'$  is the midpoint of the interval  $[\phi_1, \phi_2]$ . Since  $(p^2 - 1)(q^2 - 1) \in [\phi_1, \phi_2]$ , we have

$$|(p^2 - 1)(q^2 - 1) - N'| < \frac{1}{2}(\phi_2 - \phi_1) = \frac{(\mu - 1)^2}{2\mu}N. \quad (3)$$

Using the equation  $ed - k(p^2 - 1)(q^2 - 1) = 1$ , we get

$$\begin{aligned}\left| \frac{e}{N'} - \frac{k}{d} \right| &\leq e \left| \frac{1}{N'} - \frac{1}{(p^2 - 1)(q^2 - 1)} \right| + \left| \frac{e}{(p^2 - 1)(q^2 - 1)} - \frac{k}{d} \right| \\ &= e \frac{|(p^2 - 1)(q^2 - 1) - N'|}{N'(p^2 - 1)(q^2 - 1)} + \frac{1}{(p^2 - 1)(q^2 - 1)d} \\ &= e \frac{|(p^2 - 1)(q^2 - 1) - N'|}{N'(p^2 - 1)(q^2 - 1)} + \frac{e}{(p^2 - 1)(q^2 - 1)(k(p^2 - 1)(q^2 - 1) + 1)}\end{aligned}$$

Then, using (3), we get

$$\begin{aligned}\left| \frac{e}{N'} - \frac{k}{d} \right| &< \frac{e \frac{(\mu - 1)^2}{2\mu}N}{(\phi_1 - 1)^2} + \frac{e}{(\phi_1 - 1)^2} \\ &= \frac{e(\frac{(\mu - 1)^2}{2\mu}N + 1)}{(\phi_1 - 1)^2} \\ &= \frac{e(\frac{(\mu - 1)^2}{\mu}N + 2)}{2N^2(N - (\mu + \frac{1}{\mu}))^2}.\end{aligned}$$

Combining with the condition

$$d < \frac{N(N - (\mu + \frac{1}{\mu}))}{\sqrt{e(\frac{(\mu - 1)^2}{\mu}N + 2)}}$$

we have

$$\left| \frac{e}{N'} - \frac{k}{d} \right| < \frac{1}{2d^2}$$

So by Theorem 2,  $\frac{k}{d}$  is a convergent of the continued fraction expansion of  $\frac{e}{N'}$ , and thus, we can find the factorization of  $N$  in polynomial time. This terminates the proof.

## 6 Experiment results

### 6.1 Same bit primes

In connection with Theorem 3, we present an experimental result. We consider two RSA primes  $p$  and  $q$ , both of 512 bits, which give us a 1024-bit modulus  $N$ . The public exponent  $e$  is 2029-bit and the secret exponent  $d$  is 520-bit. We can verify that the condition  $d < \sqrt{\frac{2N^3 - 18N^2}{e}}$  is satisfied as required by Theorem 3.

```

p =1009926345 7471330007 1984329453 1837129144 5329854502
    1579205949 1972427206 6832404994 0387919056 4150845276
    8065924106 7922890816 2666245720 1616730699 1117239731 85481
q =9201052432 2086390067 1386866266 0639973895 0237269245
    6878261382 5773843108 2681621528 1513707044 8098390827
    1161420676 8781444754 1784724352 5840645389 7707377855 3491
N =9292385259 8882409829 3162082225 5290648240 2932610895
    2796745930 4569281602 2535261774 6000624542 0686015350
    5820019086 9644402726 0931963319 2462241182 6579351302
    0782053736 0725528276 5503973157 0642939798 0805113181
    1631153462 5580255681 0453116780 9138251914 8143650220
    1487534725 1944638440 3016589748 5133626528 3338290596 23064171

```

$e = 5610945328\ 2671796041\ 5829077437\ 9519142056\ 5955456458$   
 $7016831195\ 6891736518\ 8405496534\ 0274145492\ 4223492328$   
 $5533946739\ 3132613402\ 2802793166\ 8707714667\ 2017196874$   
 $2057028761\ 3007844191\ 6424007923\ 8588782842\ 5888389498$   
 $2399842002\ 8312226377\ 0538147674\ 6869931106\ 0864836776$   
 $4239757895\ 9256035850\ 8693271128\ 6691302868\ 6894236736$   
 $3858899449\ 4578621026\ 8249617044\ 5653141105\ 6506148300$   
 $2531330952\ 9779988125\ 0306197976\ 3903833872\ 8515848925$   
 $1679290735\ 1472231296\ 9415530850\ 4617614777\ 4425005468$   
 $7389111903\ 1488774848\ 5654348435\ 7158394635\ 1774298241$   
 $7614264452\ 0801690300\ 4327479471\ 1402065512\ 8418504644$   
 $3456306876\ 4694318782\ 5003370247\ 1823625867\ 3480338561\ 6726117398\ 7$

We found  $\frac{k}{d}$  at the 306th convergent of  $\frac{e}{N^2 - \frac{9}{4}N + 1}$

$k = 2230382597\ 4699469998\ 8596682209\ 1052908389\ 5670986870$   
 $3180114419\ 9709650127\ 6686575639\ 3075936836\ 3045074562$   
 $2035858758\ 0823897378\ 9584493028\ 0368787713\ 7164782033\ 05$   
 $d = 3432398830\ 0653048574\ 9095039954\ 0696608634\ 7176500716$   
 $5270469723\ 1729592771\ 5916988280\ 2606127982\ 0330727277$   
 $4886481556\ 9574042901\ 8560993999\ 8583219062\ 8701414555\ 7262923$

## 6.2 Different bit primes

In this section, we present our experimental result with Theorem 4 for the case  $\mu = 6$ . We pick two RSA primes  $p$  and  $q$  such that  $q < p < 6q$ . The prime  $p$  is 513-bit and the prime  $q$  is 511-bit. The public exponent  $e$  is 2025-bit and the secret exponent  $d$  is 520-bit. We can verify that the condition of  $d < \frac{N(N - (\mu + \frac{1}{\mu}))}{\sqrt{e(\frac{(\mu-1)^2}{\mu}N+2)}}$  is satisfied as required by Theorem 4.

$p = 2251084283\ 5399690392\ 8700279533\ 0660713571\ 6421309613$   
 $1606252397\ 0689624877\ 5416713946\ 7115639068\ 3214685532$   
 $3666826003\ 6319257221\ 4371382498\ 2817351298\ 6759602510\ 57551$   
 $q = 4309404521\ 9164250721\ 4408330391\ 2371174042\ 4509359250$   
 $1101852828\ 8585049357\ 7444908497\ 8816733294\ 5916734904$   
 $4534762750\ 3939424631\ 2488030728\ 2485748601\ 6547649010\ 5607$

$N = 9700832790\ 7021385390\ 4896466540\ 1601016697\ 9018555029$   
 $9703057514\ 5759061053\ 3697035051\ 2074639275\ 6847870272$   
 $0665951962\ 7031014290\ 6416502632\ 7630838653\ 8473467834$   
 $7894193231\ 7363670403\ 6746823112\ 7635148598\ 2833618775$   
 $9537769355\ 3973649039\ 4878199679\ 3768212843\ 1912875915$   
 $5743213282\ 9142414646\ 2589790816\ 1555160091\ 6544617094\ 24788457$

$e = 2671630446\ 5047922612\ 1252742215\ 1475559753\ 7241860791$   
 $2093132740\ 5641923165\ 9718687137\ 9309258926\ 2478467217$   
 $6240162209\ 8399694799\ 3431771506\ 3146187336\ 5667839242$   
 $9797136667\ 0172935196\ 7047211436\ 7473569771\ 6376048156$   
 $6682301440\ 4366414431\ 8903690242\ 5361894565\ 8356426103$   
 $1590986355\ 9710607287\ 9200098536\ 4620383116\ 0061469153$   
 $1021296633\ 3242652768\ 5015393792\ 5758080833\ 8675858392$   
 $6006508016\ 4569283723\ 6478429721\ 9444664288\ 7134749805$   
 $8459839679\ 1047808065\ 9191027176\ 5899435088\ 0770410882$   
 $2622162123\ 1125991843\ 0044728097\ 1298291167\ 5436520870$   
 $8063051557\ 0718647958\ 9297699102\ 8690718545\ 8180532010$   
 $5143027320\ 1226957906\ 0997422184\ 9734372569\ 9623973425\ 7519040887$

We found  $\frac{k}{d}$  at the 291th convergent of  $\frac{e}{N^2+1-\frac{(\mu+1)^2}{2\mu}N} = \frac{e}{N^2+1-\frac{49}{12}N}$

$k = 97444422179\ 7382625455\ 6522084013\ 6478311497\ 0194986256$   
 $1788936640\ 5212659933\ 2382608493\ 4392942988\ 2877863728$   
 $9298553572\ 7992595886\ 9245052334\ 7340515904\ 3019272879$   
 $d = 3432398830\ 0653048574\ 9095039954\ 0696608634\ 7176500716$   
 $5270469723\ 1729592771\ 5916988280\ 2606127982\ 0330727277$   
 $4886481556\ 9574042901\ 8560993999\ 8583219062\ 8701414555\ 7412423$

## 7 Conclusion

We have proposed an attack on three variants of the RSA cryptosystem, namely the Kuwakado-Koyama-Tsuruoka extension for singular elliptic curves, Elkamchouchi et al.'s extension of RSA to the Gaussian integer ring and Castagnos scheme. For the three extensions, we showed that the RSA modulus  $N = pq$  can be factored in polynomial time if the public exponent  $e$  is related to a suitably

small secret exponent  $d$ . The attack is based on the theory of continued fractions and can be seen as an extension of Wiener's [12] and Bunder-Tonien's [3] attacks on the RSA.

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