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ABSTRACT. We give variations on Ando's result comparing $f(B) - f(A)$ and $f(|B - A|)$ with respect to unitarily invariant norms on matrices.

This note deals with norm matricial inequalities. Our starting point is the following inequality by Ando in [1]: if A, B are positive matrices and $\|.\|$ is a unitarily invariant norm and f is an operator monotone function on $\mathbb{R}^+$ with $f(0) \geqslant 0$, then

$$\|f(B) - f(A)\| \leqslant \|f(|B - A|)\|. \tag{1}$$

The result was also obtained by Birman Koplienko and Solomyak in [3]. The inequality reverses if the reciprocal of f is operator monotone, this holds for instance if f is an increasing operator convex function with $f(0) = 0$.

In [10], it is shown that for any $p \geqslant 2$,

$$\mathrm{Tr}(B - A)(B^{p-1} - A^{p-1}) \geqslant \mathrm{Tr}|B - A|^p.$$

This has been recently extended in [5] by Dinh, Ho, Le and Vo to any operator convex function f with $f(0) = 0$:

$$\mathrm{Tr}(B - A)(f(B) - f(A)) \geqslant \mathrm{Tr}|B - A|f(|B - A|).$$

The inequality is reversed if f is non-negative operator monotone. It is naturally tempting to imagine that for any positive operator monotone function one has

$$\|(B - A)(f(B) - f(A))\| \leqslant \||B - A|f(|B - A|)\|,$$

with reversed inequality for positive operator convex functions. The aim of this note is to show that such inequalities hold. To do so, we revisit Ando's argument to see how it can be extended.

In the first section, we review basic facts on various comparisons of matrices before using them to deduce the main inequalities. We assume that the reader is familiar with matricial inequalities. We have chosen to stick with matricial inequalities but most of what is done here can be adapted to general semifinite von Neumann algebras.

1. Comparisons of matrices

We refer [2] for basic background on matricial inequalities. As usual, $\mathbb{M}_n$ is the space of matrices of size n over $\mathbb{C}$ with its usual trace Tr. We denote by $\mathbb{M}_n^+$ its subset of positive semidefinite matrices.

Given $A \in \mathbb{M}_n$, we denote by $s_i(A)$ its singular values in decreasing order. We will frequently use that if $A \in \mathbb{M}_n^+$ and $f : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ is a positive non-decreasing function then $s_i(f(A)) = f(s_i(A))$.

We recall classical orders beyond the usual one $\leqslant$ on selfadjoint matrices.

First for A, $B \in \mathbb{M}_n$, we write $A \preceq B$ if for all $1 \leqslant k \leqslant n$, $\sum_{i=1}^k s_i(A) \leqslant \sum_{i=1}^k s_i(B)$.

If we set $\|A\|_{(k)} = \sum_{i=1}^k s_i(A)$ for the Ky Fan norms, Ky Fan's principle (Theorem IV.2.2 in [2]) gives that if $A \preceq B$ iff $\|A\| \leqslant \|B\|$ for any unitarily invariant norm. This is also equivalent to the existence of a completely positive map $T : \mathbb{M}_n \to \mathbb{M}_n$ with $T(1) \leqslant 1$ and $\mathrm{Tr} \circ T \leqslant \mathrm{Tr}$ with $T(|B|) = |A|$ see [8]. Using the polar decomposition, we obtain that $A \preceq B$ iff there is a map $T : \mathbb{M}_n \to \mathbb{M}_n$, which is contractive for all unitarily invariant norms, so that $T(B) = A$. We won't use it but we recall that if $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing convex function and $A, B \in \mathbb{M}_n^+$ then $\varphi(A) \preceq \varphi(B)$ if $A \preceq B$.

Finally, for $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_N$ with $N \geqslant n$, we write $A \ll B$ if for all $1 \leqslant k \leqslant n$, $s_k(A) \leqslant s_k(B)$. Weyl's monotonicity principle, Corollary III.2.3 in [2], gives that for $A, B \in \mathbb{M}_n^+$

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with $A \leqslant B$, $A \ll B$. Thus, using the polar decomposition and diagonalization, it is easy to see that for $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_N$, $A \ll B$ iff there are contractions $C, C' \in \mathbb{M}_{N,n}$ so that $|A| = C^*|B|C$ and $A = C'^*BC$. Of course for $A, B \in \mathbb{M}_n$, $A \ll B$ implies $A \preceq B$.

Now we gather some facts about these comparisons. They must be folklore but we give a proof for completeness. For $A, B \in \mathbb{M}_n$, we write $A \oplus B$ for $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{M}_{2n}$.

Lemma 1.1. *If A, $B \in \mathbb{M}_n^+$, then $B - A \ll B \oplus A$.*

Proof. Since $B - A$ is selfadjoint, it can be written as $B - A = D_+ - D_-$ where $D_\pm \in \mathbb{M}_n^+$ and $D_+D_- = 0$. It follows that for any $1 \leqslant k \leqslant n$, $s_k(B - A) = s_k(D_+ \oplus D_-)$. Let e and f be the support projections of D_+ and D_-, then

$$0 \leqslant D_+ \oplus D_- = e(B - A)e \oplus f(A - B)f \leqslant eBe \oplus fAf.$$

The result then follows by Weyl's monotony principle as $e \oplus f$ is a contraction. □

Lemma 1.2. *Let $D \in \mathbb{M}_n^+$ and $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing functions and $A_i \in \mathbb{M}_n$ for $1 \leqslant i \leqslant d$ such that $A_i \preceq g_i(D)$, then $\sum_{i=1}^d A_i \preceq (\sum_{i=1}^d g_i)(D)$.*

Proof. This is just the triangular inequality for the norms $\|.\|_{(k)}$ combined with the fact that $\sum_i \|g_i(D)\|_{(k)} = \|\sum g_i(D)\|_{(k)}$. Indeed the g_i are non-decreasing so it yields that $s_j(\sum_i g_i(D)) = \sum_i s_j(g_i(D))$ for any $1 \leqslant j \leqslant n$. □

Lemma 1.3. *Let $D \in \mathbb{M}_n^+$ and $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing functions and $A_i \in \mathbb{M}_n$ so that $A_i \preceq g_i(D)$ for $1 \leqslant i \leqslant d$, then $\prod_{i=1}^d A_i \preceq (\prod_{i=1}^d g_i)(D)$.*

Proof. By induction, it suffices to do it for $d = 2$. We have $A_i = T_i(g_i(D))$ for some map $T_i : \mathbb{M}_n \to \mathbb{M}_n$ which is contractive for all unitarily invariant norms. Let $P_j = 1_{[s_j(D),\infty)}(D)$, since g_i is non-decreasing there are positive reals $a_{i,j}$ so that $g_i(D) = \sum_{j=1}^n a_{i,j}P_j$. Hence $A_1A_2 = \sum_{j_1,j_2} a_{1,j_1}a_{2,j_2}T_1(P_{j_1})T_2(P_{j_2})$. But $T_i(P_{j_i})$'s are contractions and

$$\|T_1(P_{j_1})T_2(P_{j_2})\|_{(k)} \leqslant \min\{\|T_1(P_{j_1})\|_{(k)}, \|T_2(P_{j_2})\|_{(k)}\} \leqslant \min\{\|P_{j_1}\|_{(k)}, \|P_{j_2}\|_{(k)}\} = \|P_{j_1}P_{j_2}\|_{(k)}.$$

This means that $T_1(P_{j_1})T_2(P_{j_2}) \preceq P_{j_1}P_{j_2}$. Noticing that $P_{j_1}P_{j_2}$ is a non-decreasing function of D and $(g_1g_2)(D) = \sum_{j_1,\dots,j_d} a_{1,j_1}a_{2,j_2}P_{j_1}P_{j_2}$, we get the conclusion by Lemma 1.2. □

Given a non-commutative polynomial in several variables

$$P(X_1, ..., X_d) = \sum_{l=0}^{m} \sum_{i_1,\dots,i_l=1}^{d} \alpha_{i_1,\dots,i_l} X_{i_1}...X_{i_l},$$

we define $|P|$ as

$$|P|(X_1, ..., X_d) = \sum_{l=0}^{m} \sum_{i_1,\dots,i_l=1}^{d} |\alpha_{i_1,\dots,i_l}| X_{i_1}...X_{i_l}.$$

Lemma 1.4. *Let $D \in \mathbb{M}_n^+$ and $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing functions and $A_i \in \mathbb{M}_n$ such that $A_i \preceq g_i(D)$ for $1 \leqslant i \leqslant d$, then for any non-commutative polynomial P of d-variables, we have*

$$P(A_1, ..., A_d) \preceq |P|(g_1(D), ..., g_d(D)).$$

Proof. This is just the combination of Lemmas 1.2 and 1.3. □

Remark 1.5. *One can extend the above lemmas in many ways. For instance, we can assume that we have a continuous sets of variables $(A_i(s))$ and replace sums $\sum_{i_1,\dots,i_l=1}^d$ by integrals against positive measures as long as the objects make sense.*

2. Main inequalities

First we rewrite Ando's proof from [1] (see also Section X in [2]). We fix $s > 0$ and consider the function on $\mathbb{R}^+$, $f_s(t) = \frac{t}{s+t} = 1 - \frac{s}{s+t}$. For convenience, we set $f_0(t) = t$. These are the basic bricks for operator monotone functions.

Lemma 2.1. *Let A, $D \in \mathbb{M}_n^+$, then $f_s(A+D) - f_s(A) \ll f_s(D)$ for any $s \geqslant 0$.*

Proof. This is obvious if $s = 0$, we assume $s > 0$.

First we have the identity, $f_s(A+D) - f_s(A) = s\big((A+s)^{-1} - (A+D+s)^{-1}\big)$. With $C = s^{1/2}(s+A)^{-1/2}$, which is a contraction, we have $f_s(A+D) - f_s(A) = Cf_s(CDC)C$. It follows that $f_s(A+D) - f_s(A) \ll f_s(CDC)$. But CDC is unitarily equivalent to $0 \leqslant D^{1/2}C^2D^{1/2} \leqslant D$. As f_s is operator monotone, we end up with $f_s(A+D) - f_s(A) \ll f_s(D^{1/2}CD^{1/2}) \leqslant f_s(D)$. □

Lemma 2.2. *Let A, $B \in \mathbb{M}_n^+$, then $f_s(B) - f_s(A) \ll f_s(|B-A|)$ for all $s \geqslant 0$.*

Proof. Put $D = B - A$ and define $D_\pm$ as above. Then $A + D_+ = B + D_-$ and by operator monotony of f_s and Lemma 1.1 since $f_s(B) - f_s(A) = f_s(B) - f_s(B + D_-) + f_s(A + D_+) - f_s(A)$,

$$f_s(B) - f_s(A) \ll \big(f_s(B + D_-) - f_s(B)\big) \oplus \big(f_s(A + D_+) - f_s(A)\big).$$

Thanks to Lemma 2.1, we get $f_s(B) - f_s(A) \ll f_s(D_-) \oplus f_s(D_+)$.

But as D_- and D_+ commute and have disjoint supports, $f_s(D_-) \oplus f_s(D_+)$ and $f_s(|D|) \oplus 0$ are unitarily equivalent in $\mathbb{M}_{2n}$ and we can conclude. □

An operator monotone function g with $g(0) = 0$ has an integral representation $g(t) = \int_{\mathbb{R}^+} f_s(t)d\mu(s)$, for some positive measure μ (that may charge 0) such that $\int_{\mathbb{R}^+} \frac{1}{1+s}d\mu(s) < \infty$. Thus Ando's result, $g(B) - g(A) \preceq |B - A|$ if $A, B \in \mathbb{M}_n^+$ follows from Lemma 2.2, as $f_s(B) - f_s(A) \ll f_s(|B - A|)$ and the extension of Lemma 1.2 to integrals.

By Lemma 1.4, we directly get

Theorem 2.3. *Let $d, e \geqslant 1$ and $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ be operator monotone functions with $g_i(0) = 0$ for $1 \leqslant i \leqslant d$ and $h_i : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing. Then for P a non-commutative polynomial of $d + e$ variables and any A, $B \in \mathbb{M}_n^+$ and any matrices C_i so that $C_i \preceq h_i(|A - B|)$ for $1 \leqslant i \leqslant e$, we have*

$$\begin{aligned}&P\big(g_1(B) - g_1(A), ..., g_d(A) - g_d(B), C_1, ..., C_e\big)\\ &\quad\preceq |P|\big(g_1(|B - A|), ..., g_d(|A - B|), h_1(|A - B|), ..., h_e(|A - B|)\big).\end{aligned}$$

Remark 2.4. *One can see directly in the proof that one just need to assume $g_i(0) \geqslant 0$. This can also be seen as applying Lemma 1.4 one more time, as $(g_i - g_i(0))(|B - A|) \ll g_i(|B - A|)$.*

The above theorem can be extended to more general objects other than polynomials and contains many particular cases. We give a few examples, assuming that $(g_i)_{i\geqslant 1}$ are operator monotone functions with $g_i(0) \geqslant 0$ and h_i are non-decreasing functions with $h(0) \geqslant 0$. For any unitarily invariant norm and any d, we have:

$$\|\prod_{i=1}^{d}(g_i(B) - g_i(A))\| \leqslant \|\prod_{i=1}^{d} g_i(|B - A|)\|, \tag{2}$$

$$\|(A - B)(g_1(A) - g_1(B))\| \leqslant \|\,|A - B|g_1(|A - B|)\|, \tag{3}$$

$$\|\sum_{i=1}^{d} h_i(|A - B|)(g_i(A) - g_i(B))\| \leqslant \|\sum_{i=1}^{d} h_i g_i(|A - B|)\|, \tag{4}$$

$$\|(A - B)\exp\big(g_1(A) - g_1(B)\big)\| \leqslant \|\,(A - B)\exp\big(g_1(|A - B|)\big)\|. \tag{5}$$

Recall that if f is operator convex on $[0, \infty)$ with $f(0) \leqslant 0$ then $t \mapsto f(t)/t$ is operator monotone on $(0, \infty)$ by [7]. In particular, if f is non-negative and $f(0) = 0$, then $f(t) = tg(t)$ for $t \in \mathbb{R}^+$ where g is operator monotone. Thus a non-negative operator convex function f on $\mathbb{R}^+$ with $f(0) = 0$ has an integral representation:

$$f(t) = \beta t + \gamma t^2 + \int_{\mathbb{R}^+} tf_s(t)d\mu(s),$$

for some (positive) measure μ (that does not charge 0) such that $\int_0^\infty \frac{1}{1+s} d\mu(s) < \infty$ and some $\beta, \gamma \geqslant 0$.

From those inequalities, one can also get results for operator convex functions, we give one example.

Theorem 2.5. *Let f be a non-negative operator convex function on $\mathbb{R}^+$ with $f(0) = 0$ and h a non-decreasing function with $h(0) \geqslant 0$, then for any A, $B \in \mathbb{M}_n^+$, we have*

$$hf(|B-A|) \preceq h(|B-A|)\big(f(B) - f(A)\big).$$

Proof. We first prove it in the case where $f(t) = \beta t + \gamma t^2 + \int_0^M t f_s(t) d\mu(s)$ for some $M \in \mathbb{R}^+$, by assumption $\beta, \gamma \geqslant 0$. Note that $t f_s(t) = t - s f_s(t)$ for $s > 0$. It follows that $f(t) = \gamma t^2 + \delta t - \int_0^M s f_s(t) d\mu(s)$ for some $\delta \geqslant 0$. The function $g(t) = \int_0^M s f_s(t) d\mu(s)$ is operator monotone on $\mathbb{R}^+$ with $g(0) = 0$. By the triangular inequality for any $n \geqslant k \geqslant 1$, with $D = B - A$:

$$\|h(|A-B|)\big(f(B) - f(A)\big)\|_{(k)} \geqslant \|h(|D|)\big(\gamma(B^2 - A^2) + \delta D\big)\|_{(k)} - \|h(|D|)\big(g(B) - g(A)\big)\|_{(k)}.$$

Let $\mathcal{E}$ be the trace preserving conditional expectation onto the (commutative) algebra generated by $D = D_+ - D_-$, we have $\mathcal{E}((A+D)^2 - A^2) = 2\mathcal{E}(A)D + D^2$. If we denote by p and q be the support projections of D_+ and D_-. As $A \geqslant 0$, $p\mathcal{E}(A) \geqslant 0$ and since $A + D_+ = B + D_-$, we get $q\mathcal{E}(A) \geqslant D_-$. Thus, $p(2\mathcal{E}(A)D + D^2) \geqslant D_+^2$ and $q(2\mathcal{E}(A)D + D^2) \leqslant -D_-^2$. Hence we arrive at $|\mathcal{E}\big(\gamma((A+D)^2 - A^2) + \delta D\big)| \geqslant \gamma D^2 + \delta|D|$, from which for any $1 \leqslant k \leqslant n$

$$\|h(|D|)\big(\gamma((A+D)^2 - A^2) + \delta D\big)\|_{(k)} \geqslant \|h(|D|)\big(\gamma D^2 + \delta|D|\big)\|_{(k)}.$$

Using inequality (4),

$$\|h(|D|)\big(f(B) - f(A)\big)\|_{(k)} \geqslant \|h(|D|)\big(\gamma D^2 + \delta|D|\big)\|_{(k)} - \|h(|D|)g(|D|)\|_{(k)}.$$

As for any $1 \leqslant i \leqslant k$, we have $s_i\big(h(|D|)\big(\gamma D^2 + \delta|D|\big)\big) - s_i(h(|D|)g(|D|)) = s_i(h(D))s_i(f(|D|))$, we get

$$\|h(|B-A|)\big(f(B) - f(A)\big)\|_{(k)} \geqslant \|h(|B-A|)f(|B-A|)\|_{(k)}.$$

The case of general f follows by approximation. □

One can also adapt the arguments to get trace inequalities as in [4]. One gets for instance from (4) that if $h : \mathbb{R} \to \mathbb{R}$ is an odd or even function non-decreasing on $\mathbb{R}^+$ with $h(0) = 0$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone, then for all $A, B \in \mathbb{M}_n^+$:

$$\Big|\mathrm{Tr}\, h(B-A)(g(B) - g(A))\Big| \leqslant \mathrm{Tr}\, hg(|B-A|).$$

The above arguments also give that if $h : \mathbb{R} \to \mathbb{R}$ is an odd function non-decreasing on $\mathbb{R}^+$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is non-negative operator convex function with $f(0) = 0$, then for all $A, B \in \mathbb{M}_n^+$:

$$\mathrm{Tr}\, h(B-A)(f(B) - f(A)) \geqslant \mathrm{Tr}\, hf(|B-A|).$$

We would like to remark that all of the above inequalities can be generalized to bounded operators with finite support on a semifinite von Neumann algebra (that one can assume to be a factor). One has to use the generalized s-numbers of [6] instead of the singular values and symmetric function spaces instead of unitarily invariant norms see [9]. We leave other possible technical extensions to the interested readers.

We conclude by noticing that (1) does not hold for general concave functions. For instance, it is false for $f(t) = \min\{t, 1\}$ and the operator or the trace norms. Indeed, (1) for the operator norm would imply that f is Lipschitz in that norm. By homogeneity and translation, this would imply that the absolute value is also Lipschitz in the operator norm on selfadjoint operators which is false.

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