Dynamic effective properties of a random configuration of cylinders in a fluid

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Forum Acusticum 2011 - Aalborg - Denmark

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Summary

The dynamic effective properties of a random medium consisting in a uniform concentration of cylindrical scatterers in an ideal fluid are looked for, with special focus on low frequencies. The effective medium is described as an isotropic viscous fluid whose mass density and dilation viscosity depend on frequency, and whose shear viscosity is nil. An explicit expression of the reflection coefficient of a harmonic plane wave incident upon the interface between the ideal fluid and the random medium may be obtained at low frequency, using the Fikioris and Waterman's approach, in two ways. In the first one, the low frequency assumption is introduced from the very beginning, while in the second one, the same hypotheses than those used by Linton et al. [J. Acoust. Soc. Am. 117 6, 2005] to calculate the effective wavenumber are used first, and, then, the low frequency assumption. In both cases, comparison of this reflection coefficient with that at the interface between the ideal fluid and the effective viscous fluid provides the effective density, which, coupled to the effective wavenumber, provides the effective dilatation viscosity. In the first case, the effective parameters found are identical to those found by Mei et al. [Phys. Rev. B 76, 2007] in a different way, while in the second case they are expressed in terms of form functions of the cylinders that reduce at low frequency to those found by Martin et al. [J. Acous. Soc. Am. 128, 2010].

PACS no. 43.20.Fn, 43.35.Bf

1. Fikioris and Waterman theory

We consider a random distribution of infinitely long identical cylindrical scatterers, areal density n_0, enclosed in an ideal fluid half-space, x>0,and an incident harmonic plane wave propagating in the x direction. The multiple scattering theory of Fikioris and Waterman [1] is used to obtain both the effective wave number and the reflection coefficient of the random medium. Letting the radius of exclusion, in the hole correction, be b, the diagonal scattering matrix of the cylinders be T (diagonal elements T_n), the effective field in the random half-space is described as a damped plane wave propagating in the x direction,

$$\varphi_E(r) = \sum_{n=-\infty}^{+\infty} A_n i^n e^{iKx}, \tag{1}$$

with K the complex effective wavenumber, and the A_n obeying the Lorentz-Lorenz law,

$$A_n = -\frac{2\pi n_0}{\left(k^2 - K^2\right)} \sum_{m=-\infty}^{+\infty} T_m A_m N_{n-m}(Kb), \tag{2}$$

as well as the extinction theorem,

$$2in_0 \frac{K+k}{\left(k^2-K^2\right)k} \sum_{n=-\infty}^{+\infty} T_n A_n = 1, \tag{3}$$

with k the wavenumber in the absence of scatterers.
Eqs.(2,3) are valid at low frequency (low ka, kb) only, for boundary effects at the fluid/random medium interface, to be negligible[1,2]. In fact, there is a boundary layer of some depth denoted as ℓ in Ref.[2] where the average field is more complicated than the coherent field in Eq.(1). Its existence is due mathematically to the fact that the configurational average of the field due to multiple scattering involves an integration on the whole area accessible to the scatterers. The integration leads to Eqs.(2,3) on the condition that the boundary between the ideal fluid and the random medium does not cross any scatterer. The depth ℓ increases thus with the radius a of the cylinders, in some complicated way that is not quite clear, so that the result of the integration is limited to small values of ka. This is the reason why we consider, from now on, the low frequency approximation. In that case, the $N_p(Kb)$ function in Eq.(2), that involves Bessel functions of ka and Kb, reduces [2,3] to

$$N_p(Kb) = -\frac{2i}{\pi}\left(\frac{K}{k}\right)^{|p|}. \tag{4}$$

Setting the determinant of the homogeneous linear system in Eq.(2) provides the dispersion equation [2,3] at low frequency,

$$\begin{aligned}\frac{K^2}{k^2} &= 1 - 4i\frac{n_0}{k^2}\sum_{n=-\infty}^{+\infty} T_n \\ &\quad -8\frac{n_0^2}{k^4}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} |m-n| T_n T_m + \ldots \\ &= 1 - 4i\frac{n_0}{k^2} f(0) \\ &\quad +8\frac{n_0^2}{\pi k^4}\int_0^{\pi} d\theta \cot\left(\frac{\theta}{2}\right)\frac{d}{d\theta}\left[f(\theta) f(-\theta)\right] + \ldots\end{aligned} \tag{5}$$

with $f(\theta)$ the far-field scattering amplitude,

$$f(\theta) = \sum_{n=-\infty}^{+\infty} T_n e^{in\theta}. \tag{6}$$

The reflection coefficient of the half-space is given by [4]

$$R = 4in_0 \frac{k-K}{2\left(k^2-K^2\right)k} \sum_{n=-\infty}^{+\infty} (-1)^n T_n A_n, \tag{7}$$

or, following the same procedure as in Ref.[5],

$$R = \frac{k-K}{k+K} \frac{1 + \sum_{n=-\infty, n\neq 0}^{+\infty} (-1)^n D_n T_n A_n}{1 + \sum_{n=-\infty, n\neq 0}^{+\infty} D_n T_n A_n}, \tag{8}$$

with D_n defined after Eq.(2) as

$$\forall n \in \mathbb{Z}^*, A_n = D_n T_0 A_0. \tag{9}$$

2. Description of the effective medium (low frequency)

In order to account for the imaginary part of the effective wavenumber, Eq.(5), the effective medium is supposed to be a non heat conducting viscous fluid in which only one type of wave (the coherent wave) may propagate, so that the shear mode is supposed to vanish, and the effective shear viscosity is set to zero. At low enough concentration indeed, contact between scatterers is highly improbable, inducing negligible shear effects. The viscous fluid is characterized then [6] by three bulk parameters, effective mass density ρ, coefficient of dilatation viscosity η, and adiabatic sound speed; in our case the latter is equal to that of the host (ideal) fluid. In a viscous fluid, the viscosity coefficient η characterizes the attenuation due to vibrating molecules. In the present case, it is related to the vibration of the cylinders, so that, contrary to the shear viscosity, it cannot be neglected, even at small concentrations.

The objective of the present paper is to determine ρ and η, given the nature of the cylinders, their concentration, as well as the characteristics (sound speed $c_0=\omega/k$ and mass density ρ_0) of the host fluid. Two equations are thus needed. Equating at first the reflection coefficient R in Eq.(8) to that of the ideal fluid/viscous fluid interface provides the effective masse density,

$$\rho = \frac{K}{k}\frac{1+R}{1-R}, \qquad (10)$$

and equating the effective wavenumber K in Eq.(5) to that of the acoustic mode [6] provides the effective viscosity

$$\eta = i\rho\frac{\omega}{k^2}\left(\frac{k^2}{K^2}-1\right). \qquad (11)$$

As recalled in Ref.[3], the transition matrix T has only a finite number of significant eigenvalues, so that the infinite series in Eqs.(2-8) extend in practice from n=-N to n=+N, with the value of N depending on frequency.

3. The effective mass density and viscosity

We look here for expansions of the effective mass density and viscosity in powers of n_0.

3.1 The low frequency Rayleigh limit

In the Rayleigh limit only three terms are needed in the infinite sums over n (N=1). Using Eq.(3), Eqs.(2,3,9) provide

$$D_1 = \frac{4in_0\dfrac{K}{k}}{k^2\left[1-\left(\dfrac{K}{k}\right)^2 - 4i\dfrac{n_0}{k^2}T_1\left[\left(\dfrac{K}{k}\right)^2+1\right]\right]}, \qquad (12)$$

and Eqs.(10,11) yield

$$\frac{\rho}{\rho_0} = \frac{1-4i\dfrac{n_0}{k^2}T_1}{1+4i\dfrac{n_0}{k^2}T_1},$$

$$\eta = -4\frac{n_0}{k^2}\frac{\rho_0\omega}{k^2}\frac{T_0+2T_1-4i\dfrac{n_0}{k^2}T_0T_1}{\left(1-4i\dfrac{n_0}{k^2}T_0\right)\left(1+4i\dfrac{n_0}{k^2}T_1\right)} \qquad ,(13)$$

along with

$$\frac{K^2}{k^2} = \frac{\left(1-4i\dfrac{n_0}{k^2}T_0\right)\left(1-4i\dfrac{n_0}{k^2}T_1\right)}{1+4i\dfrac{n_0}{k^2}T_1}. \qquad (14)$$

These results are the same as those found by Mei [7] by means of the Coherent Potential Approximation that lies upon the assumption that forward scattering in the effective medium should be nil. As this provides one equation only, Mei replaced the latter condition with two equations, obtained by the assumption that the first two terms of the forward scattering series in the effective medium vanish independently. This section shows that the only assumption needed, when using the Fikioris and Waterman theory, is that the frequency is low enough to enter the Rayleigh limit.

The static limit of Eq.(13) may be obtained from the expansion [8] of both T_0 and T_1 up to order 2 in ka,

$$T_0 = \frac{i\pi}{4}(ka)^2\left[\frac{\rho_0 c^2}{\rho_C\left(c_L^2-c_S^2\right)}-1\right],$$
$$T_1 = \frac{i\pi}{4}(ka)^2\frac{\rho_C-\rho_0}{\rho_C+\rho_0}, \qquad (15)$$

with ρ_c the mass density of the cylinders, c_L and c_S the phase velocities of the longitudinal and shear waves in them. Using these expressions and introducing the concentration of cylinders $\varphi=\pi n_0 a^2$ leads to a static effective mass density,

$$\frac{\rho}{\rho_0} = \frac{\rho_C+\rho_0+\varphi(\rho_C-\rho_0)}{\rho_C+\rho_0-\varphi(\rho_C-\rho_0)}, \qquad (16)$$

that is different from the volume averaged mass density,

$$\rho_{VA} = \varphi\rho_C + (1-\varphi)\rho_0, \qquad (17)$$

as explained long ago by Berryman [9].

In Ref.[10], Torrent *et al*. have derived the static limit of the effective mass density of a cluster of N cylinders located at given positions in an ideal fluid by equating the scattering of the N cylinders to that of an effective cylinder of mass density

$$\frac{\rho}{\rho_0} = \frac{\rho_C(\Delta+\varphi)+\rho(\Delta-\varphi)}{\rho_C(\Delta-\varphi)+\rho(\Delta+\varphi)}, \qquad (18)$$

with Δ a function of the positions of the cylinders given in Eqs.(31,9,A14) of Ref.[10]. Taking the configurational average of Eq.(18) provides again Berryman's static effective density, Eq.(16).

3.2 Intermediate (low) frequency and low concentration

We are interested here in higher frequencies than in the Rayleigh limit, that are still low enough for Eqs.(2,3) to hold. Meanwhile, the concentration is supposed low enough for $O\left(\left(n_0/k^2\right)^q\right)$ terms with q>2 to be negligible, so that we can expand all functions of the effective wavenumber in the same way as Linton and Martin [2] did to obtain the first three terms of Eq.(5). Doing so provides

$$D_n = \frac{1}{T_0} - 2i\frac{n_0}{k^2}\left[\begin{array}{l}\frac{1}{T_0}\frac{J(0)}{f(0)} + |n| + \\ \frac{1}{f(0)}\sum_{m\neq 0}|n-m|T_m - \frac{1}{T_0}\frac{f(0)+2T_0}{f(0)}I(0)\end{array}\right] \quad (19)$$

with

$$I(\alpha) = \sum_n |n| T_n e^{in\alpha} = \frac{-1}{2\pi}\int_0^{\pi}\cot\frac{\theta}{2}\left[f'(\alpha+\theta)+f'(\alpha-\theta)\right]d\theta,$$

$$J(\alpha) = \sum_n \sum_m |m-n| T_m T_n e^{in\alpha} = \frac{-1}{2\pi}\int_0^{\pi}\cot\frac{\theta}{2}\frac{d}{d\theta}\left[f(\theta)f(\alpha-\theta)+f(-\theta)f(\alpha+\theta)\right]d\theta. \quad (20)$$

The reflection coefficient is found to be

$$R = i\frac{n_0}{k^2}f(\pi) + 2\frac{n_0^2}{k^4}T_0\left[\begin{array}{l}-\frac{f(0)}{T_0}f(\pi) \\ -\frac{2f(\pi)-f(0)-T_0}{f(0)}I(0)+\frac{f(\pi)-T_0}{T_0 f(0)}J(0) \\ +\frac{f(0)-T_0}{f(0)}I(\pi)+\frac{1}{f(0)}J(\pi)\end{array}\right] \quad (21)$$

and Eqs.(10,11) finally give

$$\frac{\rho}{\rho_0} = 1-2i\frac{n_0}{k^2}\left(f(0)-f(\pi)\right) + 4\frac{n_0^2}{k^4}\left[\begin{array}{l}\frac{1}{2}\left(f(0)\right)^2-\frac{1}{2}\left(f(\pi)\right)^2 \\ +\frac{f(\pi)-T_0-f(0)}{f(0)}J(0) \\ +T_0\frac{f(0)-T_0}{f(0)}I(\pi)+\frac{T_0}{f(0)}J(\pi) \\ -T_0\frac{2f(\pi)-f(0)-T_0}{f(0)}I(0)\end{array}\right] \quad (22)$$

and

$$\eta = -4\frac{n_0}{k^2}\frac{\omega\rho_0}{k^2}f(0) - 8i\frac{n_0^2}{k^4}\frac{\omega\rho_0}{k^2}\left[2\left(f(0)\right)^2 - J(0) - f(0)\left(f(0)-f(\pi)\right)\right] \quad (23)$$

Up to first order in n_0/k^2, the effective density in Eq.(22) is identical to that derived from either Waterman and Truell's (WT) [11] or Twersky's [12] formalism. The second order term allows to account for higher concentrations, which is not possible by use of WT and Twersky's formalisms.
In the Rayleigh limit, Eqs.(22,23) provide the low concentration approximation of Eq.(13) that, in turn, gives what Martin *et al.* [13] defined as the small-φ Ament estimate of the static parameters, when the scattering coefficients T_0 and T_1 are approximated by Eqs.(15).

4. Summary and conclusion

Use of the Fikioris and Waterman's approach has allowed us to confirm the Rayleigh limit, whatever the concentration, of the effective mass density and viscosity that had been found by Mei *et al.* under seemingly stronger assumptions.
We have also obtained the second order correction in n_0/k^2 of the effective mass density of Aristegui *et al.*, as well as the expansion, up to the same order, of the effective viscosity. Performing expansions in powers of n_0/k^2 have allowed us to obtain the effective parameters in terms of the far-field scattering amplitudes of the cylinders. While *a priori* limited to low frequency, they may be valid on a wider frequency range than the Rayleigh limits.

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