DOUBLE CENTRALIZERS IN ARTIN-TITS GROUPS

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Abstract. We prove an analogue of the Centralizer Theorem in the context of Artin-Tits groups.

Introduction

To obtain information on a group G, a standard approach consists in considering subgroups and in studying how they behave in the group. In particular, one often considers the centralizer $Z_G(H)$ of a subgroup H, which is defined by

$$Z_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

This general approach naturally extends to other contexts. This is the case in the study of noncommutative algebras, where subgroups are replaced by subalgebras. Clearly, for an algebra R and a subalgebra H, the centralizer $Z_R(H)$ is also a subalgebra. In this framework, the subalgebra $Z_R(Z_R(H))$, called the *double centralizer* of H, has been considered [13, 28]. A classical result [13] is the so-called *Centralizer Theorem*, which claims that for a finite dimensional central simple algebra R over a field k and for a simple subalgebra H, one has $Z_R(Z_R(H)) = H$. Various generalizations have been obtained leading to applications [29, 8].

Coming back to the group theory framework, one is naturally led to consider the double-centralizer subgroup $Z_G(Z_G(H))$ of a subgroup H in a group G, and to address the question of a similar Centralizer Theorem. Let us denote by $DZ_G(H)$ the double centralizer of H. Obviously, when the group G has a center $Z(G)$ that is not contained in the subgroup H, the equality $DZ_G(H) = H$ can not hold. However, one may wonder whether the subgroup $DZ_G(H)$ is generated by $Z(G)$ and H. More precisely, if $Z(G) \cap H$ is trivial, one may wonder whether the equality $DZ_G(H) = Z(G) \times H$ holds. When the center of G is trivial, we recover the property of the Centralizer Theorem, namely, that the equality $DZ_G(H) = H$ holds.

As far as we know, the first Centralizer Theorem in the group theory framework has been obtained in [15] by considering the braid group on n strands and its standard parabolic subgroups. Our objective here is to address the more general case of an Artin-Tits group G and a standard parabolic subgroup H. *Artin-Tits* groups are those torsion free groups that possess a presentation associated with a Coxeter matrix. For a finite set S, a Coxeter matrix on S is a symmetric matrix $(m_{s,t})_{s,t\in S}$ where $m_{s,s} = 1$ for all s in S and $m_{s,t}$ is either a positive integer greater than 1 or ∞, for $s \neq t$. An Artin-Tits group associated with such a matrix is defined by

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the presentation

$$(1) \qquad \left\langle S \mid \underbrace{sts\ldots}_{m_{s,t}\ terms} = \underbrace{tst\ldots}_{m_{s,t}\ terms} \ ;\ \forall s,t \in S, s \neq t\ ; m_{s,t} \neq \infty \right\rangle.$$

For instance, if we consider $S = \{s_1, \ldots, s_n\}$ with $m_{s_i,s_j} = 3$ for $|i-j| = 1$ and $m_{s_i,s_j} = 2$ otherwise, we obtain the classical presentation of the braid group B_{n+1} on $n+1$ strings considered in [1]. A *standard parabolic subgroup* is a subgroup generated by a subset X of S. It turns out that such a subgroup is also an Artin-Tits group in a natural way (see Proposition 1.1 below). Artin-Tits groups are badly understood and most articles on the subject focus on particular subfamilies of Artin-Tits groups, such as Artin-Tits groups of spherical type, of RAAG type, of FC type, of large type, or of 2-dimensional type. Here again, we apply this strategy. We first consider the family of spherical type Artin-Tits groups. Seminal examples of which are braid groups. We refer to the next sections for definitions. We prove:

Theorem 0.1. *Assume that A_S is a spherical type irreducible Artin-Tits group with S as standard generating set. Let X be a proper subset of S and A_X be the standard parabolic subgroup of A generated by X. Denote by Δ the Garside element of A_S.*

(i) *If Δ lies in $DZ_{A_S}(A_X)$ but not in $Z(A_S)$, then*
$$DZ_{A_S}(A_X) = A_X \rtimes QZ(A_S)$$
where $QZ(A_S)$ acts on A_X by permutations on X.

(ii) *if Δ does not lie in $DZ_{A_S}(A_X)$ or lies in $Z(A_S)$, then*
$$DZ_{A_S}(A_X) = A_X \times Z(A_S).$$

In the above result we do not consider the cases $X = S$ and $X = \emptyset$. Indeed, for any group G one has $DZ_G(G) = G$ and $DZ_G(\{1\}) = Z(G)$ (the center of G). In the case of spherical type Artin-Tits groups, centralizers and quasi-centralizers of standard parabolic subgroups are well understood (see [14, 26, 18]). In particular, the condition on Δ is easy to check.

In the present article, we also consider Artin-Tits groups that are not of spherical type. Let us state the following conjecture.

Conjecture 0.2. Assume that A_S is an irreducible Artin-Tits group. Let A_X be a standard parabolic subgroup of A_S generated by a subset X of S. Assume that A_X is irreducible. Let A_T be the smallest standard parabolic subgroup of A_S that contains $Z_{A_S}(A_X)$.

(i) Assume that A_X is not of spherical type, then $DZ_{A_S}(A_X) = Z_{A_S}(A_T)$.

(ii) Assume that A_X is of spherical type.

   (a) If A_T is of spherical type, then
$$DZ_{A_S}(A_X) = DZ_{A_T}(A_X);$$

   (b) if A_T is not of spherical type, then
$$DZ_{A_S}(A_X) = A_X.$$

Remark 0.3. *In the above conjecture (as in Theorem 0.4) we only consider the case where A_S is irreducible. Indeed, in the general case, if $S_1, \cdots, S_k$ are the irreducible components of S, then $DZ_{A_S}(A_X)$ is the direct product of the subgroups $DZ_{A_{S_i}}(A_{X_i})$ where $X_i = S_i \cap X$ (see the first part of the proof of Theorem 2.12).*

The conjecture is supported by the following result, that we prove in Section 3:

Theorem 0.4. (i) *Conjecture 0.2 holds for all irreducible Artin-Tits groups of FC type.*
(ii) *Conjecture 0.2 holds for all Artin-Tits groups of 2-dimensional type.*
(iii) *Conjecture 0.2 holds for all Artin-Tits groups of large type.*

The reader may note in previous theorem that Point (iii) follows from Point (ii) since Artin-Tits groups of large type are of 2-dimensional type. The centralizer of a standard parabolic subgroup is well-understood in general. In particular, Conjectures 1,2 and 3 of [20] hold, for any given X in the case of the Artin-Tits groups considered in Theorem 0.4. It follows that one can read on the Coxeter graph Γ_S (see the definition in the next section) whether or not the above group A_T is of spherical type.

The reader may note that in Theorem 0.1 there is no restriction on A_X, whereas in Conjecture 0.2 we assume that A_X is irreducible. Indeed, We can extend the above conjecture to the case where X not irreducible (see Conjecture 3.4) and prove that this general conjecture holds for the same Artin-Tits groups than those considered in Theorem 0.4. However, the statement is more technical. This is why we postpone it and restrict here to the irreducible case. The remainder of this article is organized as follows. In Section 1, we introduce the necessary definitions and preliminaries. Section 2 is devoted to Artin-Tits groups of spherical type. Finally, in Section 3, we turn to the none-spherical type cases.

1. Preliminaries

In this section we introduce the useful definitions and results on Artin-Tits groups that we shall need when proving our theorems. For this whole section, we consider an Artin-Tits group A_S generated by a set S and defined by Presentation (1) given in the introduction.

1.1. Parabolic subgroups.

As explained in the previous section, the subgroups that we consider in the article are the so-called *standard parabolic subgroups*, that is those subgroups that are generated by a subset of S. One of the main reasons that explain why these subgroups are considered is that they are themselves Artin-Tits groups:

Proposition 1.1. [30] *Let X be a subset of S. Consider the Artin-Tits group A_X associated with the Coxeter matrix $(m_{st})_{s,t\in X}$. Then*
(i) *the canonical morphism from A_X to A_S that sends x to x is into. In particular, A_X is isomorphic to, and will be identified with, the subgroup of A_S generated by X.*
(ii) *If Y is another subset of S, then we have $A_X \cap A_Y = A_{X\cap Y}$.*

We have already defined the notion of the centralizer $Z_{A_S}(A_X)$ of a subgroup A_X. We recall that we denote the center $Z_{A_S}(A_S)$ of A_S by $Z(A_S)$. More generally, for a subset X of S, by $Z(A_X)$ we denote the center of the parabolic subgroup A_X. Along the way, we will also need the notions of the normalizer of a subgroup and of the quasi-centralizer of a parabolic subgroup. We recall here their definitions.

Definition 1.2. Let X be a subset of S and A_X be the associated standard parabolic subgroup.

(i) The *normalizer* of A_X in A_S, denoted by $N_{A_S}(A_X)$, is the subgroup of A_S defined by
$$N_{A_S}(A_X) = \{g \in A_S \mid g^{-1}A_X g = A_X\};$$
(ii) the *quasi-centralizer* of A_X in A_S, denoted by $QZ_{A_S}(A_X)$, is the subgroup of A_S defined by
$$QZ_{A_S}(A_X) = \{g \in A_S \mid g^{-1}Xg = X\}$$

In the sequel, we will write $QZ(A_S)$ for $QZ_{A_S}(A_S)$. There is an obvious sequence of inclusions between these subgroups:
$$Z_{A_S}(A_X) \subseteq QZ_{A_S}(A_X) \subseteq N_{A_S}(A_X).$$
But we can say more:

Theorem 1.3. [18, 19, 20] *Let A_S be an Artin-Tits group, and X be a subset of S. If A_S is of spherical type or of FC type or of 2-dimensional type, then*
$$N_{A_S}(A_X) = QZ_{A_S}(A_X) \cdot A_X.$$

This result is one of the key arguments in our proof of Theorems 0.1 and 0.4. Actually, it is conjectured in [16] that this property holds for any Artin-Tits groups.

1.2. **Families of Artin-Tits groups.** Our objective now is to introduce the various families of Artin-Tits groups that we considered in Theorems 0.1 and 0.4.

1.2.1. *Irreducible Artin-Tits groups.* First, we say that an Artin-Tits group is *irreducible* when it is not the direct product of two of its standard parabolic subgroups. Otherwise we say that it is *reducible.* Associated with the Coxeter matrix $(m_{s,t})_{s,t\in S}$ is the Coxeter graph, which is the simple labelled graph defined as it follows. Its vertex set is S and there is an edge between two distinct vertices s and t when $m_{s,t}$ is not two. The edge has label $m_{s,t}$ when $m_{s,t}$ is not 3. When $m_{s,t} = 3$, there is no label. The reader should note that there is a natural one-to-one correspondance between the standard parabolic subgroups of an Artin-Tits groups and the full subgraphs of its associated Coxeter graph. Therefore, the group A_S is irreducible if and only if the Coxeter graph Γ_S is connected. For instance the braid group on $n+1$ strings is irreducible whereas the free abelian group on two generators is not. The irreducible components of S are the vertex sets of the maximal connected full subgraphs of the Coxeter graph. Thus, A_S is the direct product of its standard parabolic subgroups generated by its irreducible components.

1.2.2. *Spherical type Artin-Tits groups.* Among Artin-Tits groups, those of spherical type are the most studied and the best understood. From Presentation (1), we obtain the presentation of the associated Coxeter group by adding the relations $s^2 = 1$ for s in S. The Artin-Tits group is said to be of spherical type when this associated Coxeter group is finite. For instance, braid groups are of spherical type as their associated Coxeter groups are the symmetric groups. Actually there is only a finite list of connected Coxeter graphs whose associated (irreducible) Artin-Tits groups are of spherical type (see [9],[3]). An almost obvious result that is of importance is that every standard parabolic subgroup of a spherical type Artin-Tits group has to be of spherical type too.

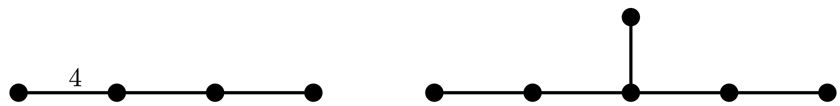


Figure 1. Coxeter graphs types $B(4)$ and $E(6)$.

1.2.3. *FC type Artin-Tits groups.* These Artin-Tits groups are built on those of spherical type. An Artin-Tits group is of FC type when all its standard parabolic subgroups whose Coxeter graphs have no edge labelled with ∞ are of spherical type. In particular, all spherical type Artin-Tits groups are of FC type. Alternatively, the family of FC type Artin-Tits groups can be defined as the smallest family of groups that contains all spherical type Artin-Tits groups and that is closed over amalgamation above a standard parabolic subgroup. For instance, the Artin-Tits group associated with the following Coxeter graph is of FC type.

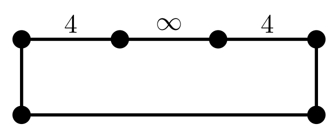


Figure 2. A Coxeter graph of FC type.

Indeed, the Artin-Tits group in Figure 2 is the amalgamation of two spherical type Artin-Tits groups of type $B(5)$ (see [2]) above a common standard parabolic subgroup, which is of type $A(4)$, that is a braid group B_5. For completeness, we mention that to each Artin-Tits group can be associated an abstract complex, the so-called *Deligne complex*. Artin-Tits groups of FC type are those for which the links of the complex vertices are flag complexes (see [6]), hence the name.

1.2.4. 2-*dimensional type Artin-Tits groups.* An Artin-Tits group is of 2-dimensional type when no standard parabolic subgroup generated by three, or more, generators is of spherical type. These groups have been considered, for instance, in [5, 7, 20].

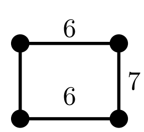


Figure 3. A 2-dimensional Coxeter graph

As for the family of FC type Artin-Tits groups, the name of this family arises from a characterising property of the Deligne complex: an Artin-Tits group is of 2-dimensional type when its Deligne complex has dimension no greater than 2 (see [5] for instance).

1.2.5. *Large type Artin-Tits groups.* Contained in the family of 2-dimensional Artin-Tits groups is the family of Artin-Tits groups of large type. An Artin-Tits group is of large type when no $m_{s,t}$ is equal to 2. Some 2-dimensional Artin-Tits groups are not of large type (see Figure 3).

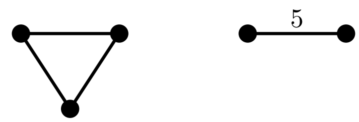


FIGURE 4. Coxeter graphs of large types $\tilde{A}(2)$ and $I(5)$.

The reader may note that a large type Artin-Tits group has to be irreducible.

1.3. **Artin-Tits monoids.** As explained above, Theorem 1.3 is one of the main ingredients in our proof. Another one is the positive monoid of an Artin-Tits group. This monoid allows to apply Garside theory. Here, we introduce only the results that we will need and refer to [10] for more details on this theory. We recall that we fix an Artin-Tits group A_S generated by a set S with Presentation (1).

Definition 1.4. The Artin-Tits monoid A_S^+ associated with A_S is the submonoid of A_S generated by S. An element of A_S that belongs to A_S^+ is called a positive element. The inverse of a positive element is called a negative element.

We gather in the following proposition several properties of Artin-Tits monoids that we will need in the sequel.

Proposition 1.5. (i) [27] *Considered as a presentation of monoid, Presentation (1) is a presentation of the monoid A_S^+.*

(ii) *When A_S is of spherical type, then*
   (a) [3, 4, 10] *Every element g in A_S can be decomposed in a unique way as $g = a^{-1}b$, with a, b positive, so that a and b have no nontrivial common left-divisors in A_S^+. Furthermore, if $c \in A_S^+$ is such that $cg \in A_S^+$, then a right-divides c in A_S^+.*
   (b) [3, 12] *There is a unique positive element Δ of minimal length so that every element g in A_S, can be decomposed as $g = h\Delta^{-n}$ where the element h is positive and $n \geq 0$. Moreover, Δ belongs to $QZ(A_S)$ and Δ^2 belongs to $Z(A_S)$.*
   (c) [3, 12] *When, moreover, A_S is irreducible then $QZ(A_S)$ is an infinite cyclic group generated by Δ. The group $Z(A_S)$ is infinite cyclic generated by Δ or by Δ^2.*

When A_S is of spherical type, the decomposition $g = a^{-1}b$ in Point (ii)(a) is called *Charney's (left) orthogonal splitting* of g. *Charney's right orthogonal splitting* $g = ab^{-1}$ is defined in a similar way. The element Δ is called the Garside element of A_S^+. For $X \subseteq S$, The submonoid A_X^+ of A_S^+ generated by X is also of spherical type. Its Garside element will be denoted by Δ_X in the remaining of the article.

By Proposition 1.5(ii)(b), the element Δ induces a permutation $\tau : S \to S$, which is either the identity or an involution, so that $\Delta s = \tau(s)\Delta$ for all s in S. By

Proposition 1.5(ii)(c), $QZ(A_S) = Z(A_S)$ if and only if τ is the identity map, that is when Δ belongs to $Z(A_S)$.

We end this section with notations and results that will be helpful in the remaining of the article (see [11, 10] for instance). For a, b in A_S^+, we write $a \preceq b$ when a left-divides b in A_S^+, that is when there exists c in A_S^+ so that $b = ac$. Similarly, we write $b \succeq a$ when a right-divides b in A_S^+. It should be noted that, when A_X is of spherical type, for $X \subseteq S$, the element Δ_X is closely related to X. On the one hand, Δ_X is the lcm of X for the left-divisibility relation $\preceq$ in A_S^+. It is also the lcm of X for the right-divisibility relation. On the other hand the set X is uniquely defined by Δ_X because X is the set of elements of S that left-divide (and right-divide) Δ_X. Moreover, given any positive word representative of Δ_X, the set of elements of S that occur in this word is X.

2. Spherical type Artin-Tits groups

In this Section we focus on spherical type Artin-Tits groups and prove Theorem 0.1.

2.1. **Artin-Tits groups of type** $E(6)$ **and** $D(2k+1)$**.** In Theorems 0.1 the description of $DZ_{A_S}(A_X)$ depends on a technical condition. Here we investigate this condition and characterize irreducible Coxeter graphs for which this condition is satisfied. For the whole section we assume that A_S is an irreducible spherical type Artin-Tits group, and X is a proper subset of S. We recall that the following properties always hold: $\Delta \in QZ(A_S)$; $\Delta^2 \in Z(A_S)$; $Z(A_S) \subseteq QZ(A_S)$; $Z(A_S) \subseteq DZ_{A_S}(A_S)$.

Proposition 2.1. *The element Δ does not belong to $Z(A_S)$ but lies in $DZ_{A_S}(A_X)$ if and only if :*

(a) *either Γ_S is of type $D(2k+1)$ and $X \supseteq \{s_2, s_{2'}, s_3\}$ (see Figure 5).*
(b) *or Γ_S is of type E_6 and $X = \{s_2, \dots, s_6\}$ (see Figure 6).*

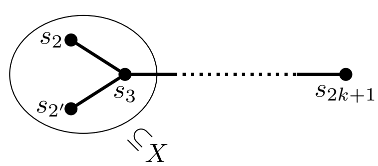


Figure 5. Γ_S of type $D(2k+1)$ and $X \supseteq \{s_2, s_{2'}, s_3\}$

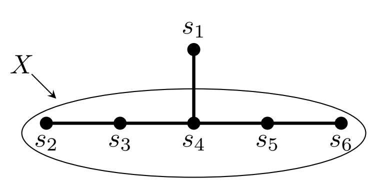


Figure 6. Γ_S of type E_6 and $X = \{s_2, \dots, s_6\}$

When proving Proposition 2.1, we will need the following lemma.

Lemma 2.2. *Assume that the element Δ does not belong to $Z(A_S)$ but lies in $DZ_{A_S}(A_X)$. Then:*

(i) *τ is the identity on $S \setminus X$, that is Δ lies in $Z_{A_S}(A_{S\setminus X})$.*
(ii) *τ is not the identity on X, that is Δ does not lie in $Z_{A_S}(A_X)$.*
(iii) *Δ stabilizes the irreducible components of X.*

Proof. (i) Let s lie in $S\setminus X$. Set $Y = X \cup \{s\}$. The elements Δ_X^2 and Δ_Y^2 lie in $Z(A_X)$ and $Z(A_Y)$, respectively by Proposition 1.5. So, they both belong to $Z_{A_S}(A_X)$ and, therefore, commute with Δ. Since $\Delta\Delta_X = \Delta_{\tau(X)}\Delta$ and $\Delta\Delta_Y = \Delta_{\tau(Y)}\Delta$, we deduce that $\tau(X) = X$ and $\tau(Y) = Y$. Using that $Y = X \cup \{s\}$, we concluded that $\tau(s) = s$ and $\Delta s = s\Delta$. Thus, Point (i) holds. Since Δ does not lie in $Z(A_S)$, the map τ is not the identity on S and (i) implies (ii). Finally, Let X_1 be an irreducible component of X. The element $\Delta_{X_1}^2$ lies in $Z(A_X)$. Indeed, it belongs to $Z(A_{X_1})$ and A_X is equal to the direct product $A_{X_1} \times A_{X\setminus X_1}$. Therefore, $\Delta\Delta_{X_1}^2 = \Delta_{X_1}^2\Delta$. On the other hand the equality $\Delta X_1 = \tau(X_1)\Delta$ imposes that $\Delta\Delta_{X_1} = \Delta_{\tau(X_1)}\Delta$ holds. This imposes in turn that $\Delta_{X_1}^2 = \Delta_{\tau(X_1)}^2$ and, finally, that $X_1 = \tau(X_1)$ (see the end of Section 1.3 for these final assertions). $\square$

Proof of Proposition 2.1. Assume that the element Δ does not belong to the center $Z(A_S)$ but lies in $DZ_{A_S}(A_X)$. Since Δ is not in $Z(A_S)$, the permutation τ is not the identity map on S. Using the classification of irreducible Artin-Tits groups [2] and well-known results on Δ [3, 12], we deduce that the type of Γ_S is one of the following:

- $A(k)$ with $k \geq 2$,
- $D(2k+1)$ with $k \geq 1$,
- E_6, or
- $I_2(2p+1)$ with $p \geq 1$.

By assumption on Δ, the assertions (i)(ii) and (iii) in Lemma 2.2 hold. Assume that Γ_S is of type $I_2(2p+1)$. In this case, the cardinality of S is two and τ, which is not the identity, has to exchange the two elements of S. This is not possible since X is proper and the permutation τ fixes each element of $S \setminus X$, by Lemma 2.2(i). Assume Γ_S is of type $A(k)$ with $k \geq 2$, (so A_S is the braid group B_{k+1}). Since X is proper, by Lemma 2.2(i), the involution τ has to fix some element of S. It follows that k is odd and the unique element of S fixed by τ is $s_{\frac{k+1}{2}}$. This imposes that we have $S \setminus X = \{s_{\frac{k+1}{2}}\}$ and Δ does not stabilize the two irreducible components $\{s_1, \ldots, s_{\frac{k-1}{2}}\}$ and $\{s_{\frac{k+3}{2}}, \ldots, s_k\}$ of X, a contradiction with Lemma 2.2(iii). So Γ_S is not of type $A(k)$. Assume that Γ_S is of type $D(2k+1)$. Then τ switches s_2 and $s_{2'}$ and fixes the other elements. Therefore, s_2 and $s_{2'}$ have to lie in the same irreducible component of X by Lemma 2.2(iii). Hence, $s_2, s_{2'}, s_3$ belong to X and we are in the case (a) of the proposition. Assume finally that Γ_S is of type E_6. We have $\tau(s_2) = s_6$. By Lemma 2.2(iii) they belong to the same irreducible component of X. Therefore, $s_2, \ldots, s_6$ lie in X. Since X is proper, it must be equal to $\{s_2, \ldots, s_6\}$ and we are in the case (b) of the proposition. Conversely, in the cases (a) and (b) of the proposition, one can verify that Δ does not belong to $Z(A_S)$ but lies in $DZ_{A_S}(A_X)$.

$\square$

2.2. **Ribbons.** The notion of a ribbon was introduced in [14] for the case of braid groups, and generalized in [26, 17]. It will be crucial to us in order to calculate the double-centralizer of a parabolic subgroup. Here we recall its definition and gather some properties that we shall need. Here, we only consider the case of spherical type Artin-Tits groups. We refer to the above references and to [10] for more details. Given an Artin-Tits presentation (1), let us first introduce two notations. For any subset X of S, we denote by $X^{\perp}$ the set of elements of S that are not in X and commute with all the elements of X. By ∂X we denote the set of elements of S that are not in X and do not commute with at least one element of X.

$$X^{\perp} = \{s \in S \setminus X \mid \forall t \in X, m_{ts} = 2\}$$

and

$$\partial X = \{s \in S \setminus X \mid \exists t \in X, m_{ts} > 2\}.$$

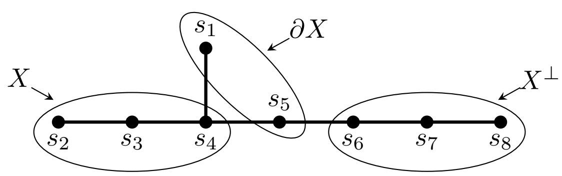


Figure 7. Example : $\partial(X)$ and $X^{\perp}$

Before reading the following definition, the reader should note that for subsets X, Y with $X \subseteq Y \subseteq S$, the element Δ_X right-divides (and also left-divides) the element Δ_Y (see the last comment in Section 1.3). Therefore $\Delta_Y \Delta_X^{-1}$ is a positive element.

Definition 2.3. (i) Let t belong to S and X be included in S. Denote by $X(t)$ the irreducible component of $X \cup \{t\}$ containing t. If t lies in X, we set $d_{X,t} = \Delta_{X(t)}$; otherwise, we set

$$d_{X,t} = \Delta_{X\cup\{t\}}\Delta_X^{-1} = \Delta_{X(t)}\Delta_{X(t)-\{t\}}^{-1}.$$

In both cases, there exist Y and t' with $Y \cup \{t'\} = X \cup \{t\}$ and $Y(t') = X(t)$ so that $Y d_{X,t} = d_{X,t} X$. The element $d_{X,t}$ is called a *positive elementary (Y, X)-ribbon.*

(ii) For $X, Y \subseteq S$, we say that $g \in A_S^+$ is a positive (Y, X)-ribbon if $Yg = gX$.

For instance, consider the Artin-Tits group A_S of type E_8, whose graph is considered in Figure 7. One has $d_{X,s_5} = s_2 s_3 s_4 s_5$, and for any t in $X^{\perp}$ one has $d_{X,t} = t$. Moreover Δ is a positive (X, X)-ribbon. In the sequel, we say that an element of A_S^+ is a positive $(\cdot, X)$-ribbon when it is a positive (Y, X)-ribbon, for some Y. Similarly we say that an element is a positive $(Y, \cdot)$-ribbon when it is a positive (Y, X)-ribbon for some X.

The connection between positive ribbons and elementary ones appears in the following result:

Proposition 2.4. *Assume that A_S is a spherical type Artin-Tits group and g lies in A_S^+. The element g is a positive (Y, X)-ribbon if and only if g can be decomposed as $g = g_n \cdots g_1$ where each g_i is a positive elementary (X_i, X_{i-1})-ribbon, with $X_0 = X$ and $X_n = Y$.*

In the remainder of the article, we need the notion of a *reduced* element. For $X \subseteq S$, we say that a positive element g is $(X, \cdot)$-reduced when no element of X left-divides g in A_S^+. This is equivalent to say that no element of A_X^+ left-divides g. Similarly we say that a positive element g is $(\cdot, X)$-reduced when no element of X right-divides g in A_S^+. Finally, for $X, Y \subseteq S$, the element g is said to be (X, Y)-reduced if it is both $(X, \cdot)$-reduced and $(\cdot, Y)$-reduced.

Proposition 2.5. *Assume that A_S is a spherical type Artin-Tits group. Let X be included in S and u lies in A_S^+. Let $\varepsilon \in \{1, 2\}$ be such that Δ_X^ε lies in $Z(A_X)$.*

(i) *Assume that u is a positive (Y, X)-ribbon for some $Y \subseteq S$.*
   (a) $\Delta_Y u = u \Delta_X$.
   (b) *Assume that t belongs to S. Then,*
$$u \succeq t \;\Leftrightarrow\; u \succeq d_{X,t}.$$
(ii) *If $u\Delta_X^\varepsilon u^{-1}$ is a positive element, then there exists $Y \subseteq S$ such that (a) $u\Delta_X^\varepsilon u^{-1} = \Delta_Y^\varepsilon$; (b) $uA_Xu^{-1} = A_Y$ and (c) Coxeter graphs Γ_X and Γ_Y are isomorphic. Moreover, if u is $(\cdot, X)$-reduced, then u is a positive (Y, X)-ribbon, that is $Yu = uX$.*

The above results are not all explicitly stated in [26, 17] but are variations of results contained there. The second part of (ii) is stated in [17, Lemma 2.2] and implies the first part of (ii)(see also [26, Lemma 5.6]). Point (i) is shown in the proof of [17, Lemma 2.2] (see [26, Lemma 5.6] for details). For point (i)(b), see also [21, Example 3.14].

The *support* of a word on S is the set of letters that are involved in this word. It follows from the presentation of A_S^+ that two words on S representing the same element in A_S^+ have the same support. So the *support* of an element of A_S^+ is well-defined. In the sequel, by $Supp(g)$ we denote the support of an element g in A_S^+.

Lemma 2.6. *Assume that A_S is a spherical type Artin-Tits group. Let $X \subsetneq S$ be such that Γ_X is connected. Let t lie in ∂X. Then*
$$Supp(d_{X,t}) = X \cup \{t\}.$$

Proof. By assumption t is not in X, so $d_{X,t} = \Delta_{X\cup\{t\}}\Delta_X^{-1}$ and $Supp(d_{X,t})$ is included in $X \cup \{t\}$. Let us show the converse inclusion. By Proposition 2.5 (i), we have $d_{X,t} = v_0 t$ for some v_0 in A_S^+; then t belongs to the support of $d_{X,t}$. Let s be in X. Set $s_0 = t$. Since X is connected and t belongs to ∂X, there exists a finite sequence $s_1, \dots, s_n$ of X such that $s_n = s$ and $m_{s_i,s_{i+1}} \neq 2$ for all $i \geq 0$. We assume that the sequence is chosen so that n is minimal. Assume that $d_{X,t} = v_i s_i \cdots s_0$ for some $0 \leq i < n$ with v_i in A_S^+. Since $Y d_{X,t} = d_{X,t} X$ for some $Y \subseteq X \cup \{t\}$, we can write $v_i s_i \cdots s_0 s_{i+1} = s'_{i+1} v_i s_i \cdots s_0$ for some s'_{i+1} in $X \cup \{t\}$. By minimality of n, we have $m_{s_j,s_{i+1}} = 2$ for any $j < i$. So $v_i s_i s_{i+1} s_{i-1} \cdots s_0 = s'_{i+1} v_i s_i \cdots s_0$ and $v_i s_i s_{i+1} = s'_{i+1} v_i s_i$. This imposes that $v_i s_i s_{i+1} = s'_{i+1} v_i s_i = v' \underbrace{\cdots s_{i+1} s_i s_{i+1}}_{m \text{ terms}}$ with $m = m_{s_i,s_{i+1}}$ and v' in A_S^+. Indeed, the element $\underbrace{\cdots s_{i+1} s_i s_{i+1}}_{m \text{ terms}}$, which is equal to $\underbrace{\cdots s_i s_{i+1} s_i}_{m \text{ terms}}$, is the left lcm of s_i and s_{i+1} for the right divisibility relation (see

[12, 3, 10] for instance). This imposes in turn that we can write $v_i = v_{i+1}s_{i+1}$, where $v_{i+1} = v'\underbrace{\cdots s_{i+1}s_is_{i+1}}_{m-2\ terms}$, and $d_{X,t} = v_{i+1}s_{i+1}s_i\cdots s_0$. Then, we obtain step-by-step that $d_{X,t}$ can be decomposed as $v_ns_n\cdots s_0$. Hence, s belongs to the support of $d_{X,t}$ for any s in X. As a consequence, $X\cup\{t\}$ is included in $Supp(d_{X,t})$. □

Lemma 2.7. *Let $u\in A_S^+$ and $s\in S$. Denote by $u_2^{-1}v_1$ the left orthogonal splitting of the element $u^{-1}su$. There exists u_1 in A_S^+ and s_1 in S so that $u = u_1u_2$, $v_1 = s_1u_2$. Moreover, u_1 is a positive $(\{s\},\{s_1\})$-ribbon.*

Proof. By [22, Theorem 1] there exists u_1 in A_S^+ and s_1 in S so that $u = u_1u_2$ and $v_1 = s_1u_2$. Moreover, applying [22, Lemma 2.3], a straightforward induction on the length of u proves that u_1 is a positive $(\{s\},\{s_1\})$-ribbon. □

2.3. **The proof of Theorem 0.1.** In this section we prove Theorem 0.1. The proof needs two preliminary results, namely Lemma 2.9 and Proposition 2.8. The latter is the main argument. Proposition 2.8 is proved here; the proof of Lemma 2.9 is postponed to the next section.

Proposition 2.8. *Let A_S be an irreducible Artin-Tits group of spherical type. Let X be a proper subset of S. Let $b\neq 1$ be a positive $(\cdot, X\cup X^\perp)$-ribbon that is $(\cdot, X)$-reduced. Suppose further that $b\Delta_Y^{\varepsilon(Y)}b^{-1}$ is a positive element for all $Y\subseteq S$ containing X, where $\varepsilon(Y)\in\{1,2\}$ is minimal such that $\Delta_Y^{\varepsilon(Y)}$ belongs to $Z_{A_S}(A_X)$. Then there exists $n\in\mathbb{N}^*$ so that*

$$b = \Delta^n\Delta_X^{-n}$$

Note that Δ_X right-divides Δ in A_S^+ and $\Delta\Delta_X = \Delta_{\tau(X)}\Delta$ by Proposition 1.5. So for any positive integer n the element $\Delta^n\Delta_X^{-n}$ is a positive element.

Lemma 2.9. *Under the assumptions of Proposition 2.8, all the elements of $S\setminus X$ right-divide b.*

Proof of Proposition 2.8. We have $\Delta_X\succeq s$ for all $s\in X$ and, by Lemma 2.9, we have $b\succeq s$ for all $s\in S\setminus X$. Since, by assumption, $b\Delta_X\succeq b$, we get that $b\Delta_X\succeq s$ for all $s\in S$. Thus $b\Delta_X\succeq\Delta$ and, therefore, $\Delta\Delta_X^{-1}$ right-divides b in A_S^+. Let $k\in\mathbb{N}^*$ be maximal such that $\Delta^k\Delta_X^{-k}$ right-divides b. Write $b = d\Delta^k\Delta_X^{-k}$ with $d\in A_S^+$. We show that $d = 1$. This will prove the proposition. For the remainder of the proof, for $Z\subseteq S$ and $j\in\mathbb{N}$, we set $Z_j = \tau^j(Z)$. Then $\Delta^kX = X_k\Delta^k$ and, by Proposition 2.5, we have $\Delta^k\Delta_X = \Delta_{X_k}\Delta^k$. Moreover, Δ_X is a positive (X,X)-ribbon. Then $\Delta^k\Delta_X^{-k}$ is a positive (X_k,X)-ribbon. For the remainder of the proof, when s lies in X_k, we denote by s_X the element of X so that $s\Delta^k\Delta_X^{-k} = \Delta^k\Delta_X^{-k}s_X$.

Assume that $d = us$ with s in X_k and u positive. Then $b = us\Delta^k\Delta_X^{-k} = u\Delta^k\Delta_X^{-k}s_X$. This is not possible, since b is $(\cdot,X)$-reduced. Hence, d is $(\cdot,X_k)$-reduced. We now prove that d is a positive $(\cdot, X_k\cup X_k^\perp)$-ribbon. Let s lie in X_k. We have $s\Delta^k\Delta_X^{-k} = \Delta^k\Delta_X^{-k}s_X$. But, b is a positive $(\cdot,X)$-ribbon. Therefore there exists s' in S so that $bs_X = s'b$. Hence $ds\Delta^k\Delta_X^{-k} = d\Delta^k\Delta_X^{-k}s_X = s'd\Delta^k\Delta_X^{-k}$. We deduce that the equality $ds = s'd$ holds. As this is so for every element s of X_k, we deduce that d is a positive $(\cdot,X_k)$-ribbon. Let s lie in $X_k^\perp$. For every t in X, the element $\tau^k(t)$ lies in X_k and, therefore, $m_{\tau^k(t),s} = 2$. But the involution τ induces an automorphism of the Coxeter graph associated with the presentation of A_S. It follows that for every t in X, we have $m_{t,\tau^k(s)} = m_{\tau^k(t),s} = 2$. Hence, $\tau^k(s)$ belongs to $X^\perp$. But b is a positive $(\cdot, X^\perp)$-ribbon, then $b\tau^k(s) = s'b$ for some $s' \in S$. We obtain that $ds\Delta^k\Delta_X^{-k} = d\Delta^k\Delta_X^{-k}\tau^k(s) = s'd\Delta^k\Delta_X^{-k}$, and, therefore, $ds = s'd$. As this is so for every element of $X_k^\perp$, we deduce that d is a positive $(\cdot, X_k^\perp)$-ribbon. Gathering the two results we get that d is a positive $(\cdot, X_k \cup X_k^\perp)$-ribbon.

Let Y be included in S and contain X_k. Consider the minimal positive integer $\eta(Y)$ such that $\Delta_Y^{\eta(Y)}$ belongs to $Z_{A_S}(A_{X_k})$. The involution τ^k exchanges X and X_k and exchanges Y and Y_k. It follows, firstly, that the inclusion $X_k \subseteq Y$ implies the inclusion $X \subseteq Y_k$ and, secondly, that τ^k sends A_{X_k} and Δ_Y to A_X and Δ_{Y_k}, respectively, with $\eta(Y) = \varepsilon(Y_k)$. Thus, $\Delta_{Y_k}^{\eta(Y)}$ belongs to $Z_{A_S}(A_X)$ with $\eta(Y) = \varepsilon(Y_k)$. Then, by assumption, we have $b\Delta_{Y_k}^{\eta(Y)} = ub$, for some u in A_S^+. Since $b\Delta_{Y_k}^{\eta(Y)} = d\Delta^k\Delta_X^{-k}\Delta_{Y_k}^{\eta(Y)} = d\Delta^k\Delta_{Y_k}^{\eta(Y)}\Delta_X^{-k} = d\Delta_Y^{\eta(Y)}\Delta^k\Delta_X^{-k}$ and $ub = ud\Delta^k\Delta_X^{-k}$ we obtain that $d\Delta_Y^{\eta(Y)} = ud$. As a consequence, replacing b and X by d and X_k, respectively, we can repeat the first argument of the proof and deduce that $d = d_1\Delta\Delta_{X_k}^{-1}$ for some d_1 in A_S^+. But this leads to a contradiction to the maximality of k, since we get $b = d\Delta^k\Delta_X^{-k} = d_1\Delta\Delta_{X_k}^{-1}\Delta^k\Delta_X^{-k} = d_1\Delta^{k+1}\Delta_X^{-(k+1)}$. Hence $d = 1$ and $b = \Delta^k\Delta_X^{-k}$. □

We turn now to the proof of Theorem 0.1.

Proof of Theorem 0.1. Let u lie in $DZ_{A_S}(A_X)$. The inclusion $Z_{A_X}(A_X) \subseteq Z_{A_S}(A_X)$ obviously holds. We deduce the inclusion $DZ_{A_S}(A_X) \subseteq Z_{A_S}(Z_{A_X}(A_X))$ and that u belongs to $Z_{A_S}(Z_{A_X}(A_X))$. Thanks to Proposition 2.5 and Theorem 1.3, we can write $u = y \cdot z$, with $yX = Xy$ and $z \in A_X$. Write (see Proposition 1.5) $y = \Delta^{-2m}h$ with h in A_S^+, and decompose h as $h = abc$, with $a, c \in A_X^+$ and b being (X, X)–reduced. Since $yX = Xy$ and Δ^2 is in $Z(A_S)$, we have $hX = Xh$ and so $h\Delta_X = \Delta_X h$. Using that $h = abc$ with a, c in A_X^+ and that Δ_X^2 lies in $Z(A_X)$, we deduce that $b\Delta_X^2 = \Delta_X^2 b$. The element b is $(\cdot, X)$-reduced, then by Proposition 2.5, we have $bX = Xb$. It follows that there exists $z' \in A_X$ such that $bcz = z'b$. Set $x = az'$. Then, x belongs to A_X and $u = yz = \Delta^{-2m}hz = \Delta^{-2m}abcz = \Delta^{-2m}az'b = \Delta^{-2m}xb$. If $b = 1$ then u lies in $Z(A_S) \cdot A_X$, and therefore in $QZ(A_S) \cdot A_X$. Suppose $b \neq 1$. By its definition, the set $X^\perp$ is included in $Z_{A_S}(A_X)$. Therefore, for all s in $X^\perp$ we have $us = su$ and $s\Delta^{-2m}xb = \Delta^{-2m}xsb$. By cancellation, we obtain $bs = sb$ for all $s \in X^\perp$. So, b is a positive $(\cdot, X \cup X^\perp)$-ribbon.

Now, let Y be included in S and containing X. Set $\varepsilon(Y)$ in $\{1, 2\}$ be minimal such that $\Delta_Y^{\varepsilon(Y)}$ lies in $Z_{A_S}(A_X)$. Then, $u\Delta_Y^{\varepsilon(Y)} = \Delta_Y^{\varepsilon(Y)}u$ and, as before, we get $b\Delta_Y^{\varepsilon(Y)} = \Delta_Y^{\varepsilon(Y)}b$. As a consequence we have $b\Delta_Y^{\varepsilon(Y)}b^{-1}$ is positive. By Proposition 2.8, we deduce there exists n in $\mathbb{N}^*$ so that $b = \Delta^n\Delta_X^{-n}$. Thus, we get $u = \Delta^{-2m}x\Delta^n\Delta_X^{-n}$.

Assume, first, that Δ lies in $DZ_{A_S}(A_X)$ and does not belong to $Z(A_S)$. Then, by Lemma 2.2, we have $\Delta X = X\Delta$ and $\tau^n(x)$ belongs to A_X. Therefore, we can write $u = \Delta^{-2m+n} \cdot \tau^n(x)\Delta_X^{-n}$. Since Δ belongs to $QZ(A_S)$ and $\tau^n(x)$ belongs to X, we deduce that u belongs to $QZ(A_S) \cdot A_X$. So $DZ_{A_S}(A_X)$ is included in $QZ(A_S) \cdot A_X$. Conversely, Δ generates $QZ(A_S)$. So, if Δ lies in $DZ_{A_S}(A_X)$ then we have $QZ(A_S) \cdot A_X \subseteq DZ_{A_S}(A_X)$. Therefore, the latter inclusion is actually an equality. Moreover we have $QZ(A_S) \cdot A_X = A_X \cdot QZ(A_S)$, since Δ belongs to $QZ_{A_S}(A_X)$ by the above argument.

Assume, secondly, that either Δ does not lie in $DZ_{A_S}(A_X)$ or Δ belongs to $Z(A_S)$. Since u lies in $DZ_{A_S}(A_X)$, for every w in $Z_{A_S}(A_X)$ we have $wu = uw$ and, therefore, $\Delta^{-2m}xw\Delta^n\Delta_X^{-n} = \Delta^{-2m}x\Delta^n w\Delta_X^{-n}$. This imposes $\Delta^n w = w\Delta^n$ for every w in $Z_{A_S}(A_X)$. In other words Δ^n lies in $DZ_{A_S}(A_X)$ too. Since either Δ does not lie in $DZ_{A_S}(A_X)$ or Δ belongs to $Z(A_S)$, we deduce that either n is even or Δ belongs to $Z(A_S)$. In any case, Δ^n belongs to $Z(A_S)$. Since we have $u = \Delta^{-2m+n}x\Delta_X^{-n}$, we deduce that u lies in $Z(A_S) \cdot A_X$. So the inclusion $DZ_{A_S}(A_X) \subseteq Z(A_S) \cdot A_X$ holds.

Since $X \neq S$ and $Supp(\Delta) = S$, the element Δ^m does not belong to A_X except if $m = 0$. We deduce that $A_X \cap QZ(A_S) = A_X \cap Z(A_S) = \{1\}$. We get the semi-direct product $A_X \ltimes QZ(A_S)$ in the first case, and the direct product $A_X \times Z(A_S)$ in the second case. □

2.4. **The proof of Lemma 2.9.** Here we focus on the proof of Lemma 2.9. This proof is technical and, to help the reader, we decompose it in 3 steps, namely Lemma 2.10, Lemma 2.11 and the final argument.

Lemma 2.10. *Under the assumptions of Proposition 2.8, if t lies in ∂X then*

$$b \not\succeq t \;\Leftrightarrow\; bt = t'b \text{ for some } t' \in S.$$

Proof. Assume that $b \not\succeq t$. Set $Y = X \cup \{t\}$. Under the assumptions of Proposition 2.8, we have $b\Delta_Y^{\varepsilon(Y)} \succeq b$. By Proposition 2.5, we deduce there exists some subset Y' of S such that $b\Delta_Y^{\varepsilon(Y)}b^{-1} = \Delta_{Y'}^{\varepsilon(Y)}$ and $bA_Yb^{-1} = A_{Y'}$. On the other hand, the element b is a positive (X', X)-ribbon for some subset X' of S. It follows that X' is included in $A_{Y'}$ and, therefore, in Y'. Now, the sets X' and Y' have the same cardinality as X and Y, respectively. Then there exists t' in Y' so that $Y' = X' \cup \{t'\}$. We are going to prove that $bt = t'b$. By Lemma 2.7, we can decompose b as $b = b_1b_2$ with b_2 in A_S^+ and b_1 a positive $(\{t'\}, \{t''\})$-ribbon for some t'' in S, so that the left orthogonal splitting of $b^{-1}t'b$ is $b_2^{-1}t''b_2$. By the above argument $b^{-1}t'b$ lies in A_Y, so t'' has to lie in Y and b_2 has to lie in A_Y^+. But b is $(\cdot, Y)$-reduced. Indeed, we assumed that b is $(\cdot, X)$-reduced and that $b \not\succeq t$. This imposes $b_2 = 1$, $b = b_1$ and $bt'' = t'b$ for some t'' in Y. Finally we already have $X'b = bX$. Since t' does not belong to X', It follows that t'' cannot lie in X. Thus $t'' = t$ and we are done.

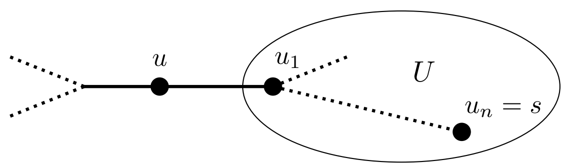
Conversely, Assume that btb^{-1} is positive, then b is a positive $(\cdot, \{t\})$-ribbon. Since it is a positive $(\cdot, X)$-ribbon, it is also a positive $(\cdot, Y)$-ribbon. By $Y(t)$ denote the irreducible component of Y that contains t. Since t lies in $\partial(X)$, the set $Y(t)$ contains some element of X. By Proposition 2.5 (i)(b), if t is a right-divisor of b then $d_{Y,t}$ is a right-divisor of b too. but t lies in Y, therefore $d_{Y,t} = \Delta_{Y(t)}$. So all the elements of $Y(t)$ are right-divisors of b. But b is $(\cdot, X)$-reduced, a contradiction. Thus t does not right-divide b. □

Note that we showed the above result without using that b is a positive $(\cdot, X^{\perp})$-ribbon. This hypothesis is then useless for Lemma 2.10.

Lemma 2.11. *Under the assumptions of proposition 2.8, we have*

$$Supp(b) = S.$$

Proof. By assumption $b \neq 1$, so its support is not empty. Assume by contradiction that $Supp(b) \neq S$. Let U be an irreducible component of $Supp(b)$. Fix u in ∂U and set $V = Supp(b) \setminus U$. By hypothesis u does not lie in $Supp(b)$. Then, u does not right-divide b. The element b is a positive $(\cdot, X \cup X^{\perp})$-ribbon. So, if u belongs to $X \cup X^{\perp}$, then bub^{-1} lies in A_S^+. Otherwise u lies in ∂X and, by Lemma 2.10, bub^{-1} lies in A_S^+ too. Now, the set U is an irreducible component of $Supp(b)$, and V is the union of the other components of $Supp(b)$. Then, each element of U commutes with each element of V and we can write $b = b_2 b_1$ with $b_1 \in A_U^+$, and $b_2 \in A_V^+$. Since U is included in $Supp(b)$, we have $b_1 \neq 1$. Write $b_1 = b_1' s$ with $s \in U$. Since $U \cup \{u\}$ is irreducible, there exists $u_1, \ldots, u_n \in U$ such that $u_0 = u$, $u_n = s$ and $m_{u_i,u_{i+1}} > 2$. Up to replacing s by some u_i with $i < n$, we can assume that b has no right-divisor among $u_1, \ldots, u_{n-1}$.



Set $U' = \{u, u_1, \ldots, u_{n-1}\}$. Let u_i lie in U'. If u_i does not lie in $X \cup X^{\perp}$, then u_i belongs to ∂X and, by Lemma 2.10, bu_ib^{-1} lies in S. On the other hand b is a positive $(\cdot, X \cup X^{\perp})$-ribbon. Therefore b is a positive $(\cdot, U')$-ribbon. By its definition the graph $\Gamma_{U'}$ is connected, the element u_n lies in $\partial U'$ and right-divides b. Then by Proposition 2.5, the positive elementary ribbon d_{U',u_n} right-divides b. Applying Lemma 2.6, we get that U' is contained in the support of b. Therefore, u belongs to $Supp(b)$, a contradiction. Hence, $Supp(b) = S$. □

We are now ready to prove Lemma 2.9.

proof of Lemma 2.9. Let s lie in $S \setminus X$, and set $Y = S \setminus \{s\}$. Write $b = b_1 b_2$ with b_2 in A_Y^+ and b_1 $(\cdot, Y)$-reduced. By Lemma 2.11, we have $Supp(b) = S$. Since b_2 lies in A_Y^+, it follows that $b_1 \neq 1$. In addition, b_1 is $(\cdot, Y)$-reduced. Then, s has to right-divide b_1. We have $b\Delta_Y^2 b^{-1} = b_1 \Delta_Y^2 b_1^{-1}$. According to the assumptions of Proposition 2.8, we have $b\Delta_Y^2 = zb$ for some z in A_S^+. Indeed, if $\varepsilon(Y) = 1$ then $b\Delta_Y = z_1 b$ for some $z_1 \in A_S^+$. Therefore $b\Delta_Y^2 = z_1 b \Delta_Y = z_1^2 b$. By Proposition 2.5, we deduce that b_1 is a (Y', Y)-ribbon for some $Y' \subseteq S$ and $b = b_1 b_2 = b_2' b_1$ with $b_2' \in A_{Y'}^+$. Since s right-divides b_1, it also has to right-divide b. □

2.5. **When Γ_S is not connected.** In Theorem 0.1 we consider irreducible Artin-Tits groups of spherical type. Here we extend the theorem to any spherical type Artin-Tits group.

Theorem 2.12. *Let A_S be an Artin-Tits group of spherical type. Denote the irreducible components of S by $S_1, \ldots, S_n$. Let A_X be a standard parabolic subgroup of A_S and set $X_i = X \cap S_i$ for all i. Set*

$$I = \{1 \leq i \leq n \mid X_i \neq S_i, \Delta_{S_i} \in DZ_{A_{S_i}}(A_{X_i}) \text{ and } \Delta_{S_i} \notin Z(A_{S_i})\}.$$

$$J = \{1 \leq i \leq n \mid X_i \neq S_i, \text{ and } i \notin I\}.$$

Finally, set $S_I = \bigcup_{i \in I} S_i$ and $S_J = \bigcup_{i \in J} S_i$. Then we have

$$DZ_{A_S}(A_X) = (A_X \rtimes QZ(A_{S_I})) \times Z(A_{S_J}).$$

Proof. Consider a direct product of groups $G = G_1 \times \cdots \times G_n$ and a subgroup H of G that is the direct product $H_1 \times \cdots \times H_n$ of its subgroups H_i, where $H_i = H \cap G_i$ for i in $1, \cdots, n$. We have

$$Z_G(H) = Z_{G_1}(H_1) \times \cdots \times Z_{G_n}(H_n)$$

Since we again have a direct product of subgroups of each G_i we get

$$DZ_G(H) = DZ_{G_1}(H_1) \times \cdots \times DZ_{G_n}(H_n)$$

Here, $A_S = A_{S_1} \times \cdots \times A_{S_n}$ and $A_X = A_{X_1} \times \cdots \times A_{X_n}$, with $A_X \cap A_{S_i} = A_{X_i}$, by Proposition 1.1. By Theorem 0.1, if i lies in I, then $DZ_{A_{S_i}}(A_{X_i}) = A_{X_i} \rtimes QZ(A_{S_i})$; if i lies in J then $DZ_{A_{S_i}}(A_{X_i}) = A_{X_i} \times Z(A_{S_i})$. In addition, if i is neither in I nor in J, then $X_i = S_i$ and $DZ_{A_{S_i}}(A_{X_i}) = A_{X_i}$. So, we deduce that

$$DZ_{A_S}(A_X) = \prod_{i=1}^{n} DZ_{A_{S_i}}(A_{X_i}) =$$

$$\prod_{i\in I}(A_{X_i} \rtimes QZ(A_{S_i})) \times \prod_{j\in J}(A_{X_j} \times Z(A_{S_j})) \times \prod_{k\notin I\cup J} A_{X_k} =$$

$$\left(\prod_{k=1}^{n} A_{X_k} \rtimes \prod_{i\in I} QZ(A_{S_i})\right) \times \prod_{j\in J} Z(A_{S_j}).$$

But $\prod_{k=1}^{n} A_{X_k} = A_X$, $\prod_{i\in I} QZ(A_{S_i}) = QZ(A_{S_I})$ and $\prod_{j\in J} Z(A_{S_j}) = QZ(A_{S_J})$. $\square$

2.6. **Application to the subgroup conjugacy problem.** Given a group G and a subgroup H of G, the subgroup conjugacy problem for H is solved by finding an algorithm that determines whether any two given elements of G are conjugated by an element of H. In this section, we focus on Artin-Tits groups of type B or D and use Theorem 0.1 and [25, Theorem 1.1] to reduce the subgroup conjugacy problem for their irreducible standard parabolic subgroups to an instance of the simultaneous conjugacy problem. We follow the strategy used in [15] to solve the subgroup conjugacy problem for irreducible standard parabolic subgroups of an Artin-Tits group of type A. The simultaneous conjugacy problem is solved for Artin-Tits groups of type A in [23] (see also [24]), but the result and its proof can be generalized verbatim to all Artin-Tits groups of spherical type, in particular to Artin-Tits groups of type B or type D . Hence, we obtain a solution to the subgroup conjugacy problem for irreducible standard parabolic subgroups of Artin-Tits groups of type B and D.

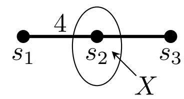
Let us recall [25, Theorem 1.1] and [15, Theorem 2.13].

Theorem 2.13 ([25], Theorem 1.1)**.** *Let A_S be an Artin-Tits group of spherical type such that $\Gamma_S = A_k$ $(k \geq 1)$, $\Gamma_S = B_k$ $(k \geq 2)$ or $\Gamma_S = D_k$ $(k \geq 4)$. Let $X \subseteq S$ such that Γ_X is connected. Then $Z_{A_S}(A_X)$ is generated by*

$$X^{\perp} \cup \{\Delta_Y \in Z_{A_S}(A_X) \mid X \subseteq Y\} \cup \{\Delta_Y \Delta_{Y'} \in Z_{A_S}(A_X) \mid X \subseteq Y, X \subseteq Y'\}.$$

Note that in the third set, we can restrict the pair (Y, Y') to those so that neither Δ_Y nor $\Delta_{Y'}$ belong to $Z_{A_S}(A_X)$. In the sequel, we denote the obtained generating set by $\Xi(X)$.

Example 2.14. Consider $S = \{s_1, s_2, s_3\}$ with A_S of type B_3 as below. Set $X = \{s_2\}$.



We have $X^\perp = \emptyset$ and $Z_{A_S}(A_X)$ is generated by $\Xi(X) = \{s_2, \Delta_{\{s_1,s_2\}}, \Delta_S, \Delta^2_{\{s_2,s_3\}}\}$.

Theorem 2.15 ([15], Theorem 2.13)**.** *Let G be a group and H be a subgroup such that $DZ_G(H) = Z(G) \cdot H$. Suppose further that $Z_G(H)$ is generated by a set $\{g_1, \ldots, g_n\}$. Then for $x, y \in G$, the following are equivalent:*

(i) *there exists $c \in H$ such that $y = c^{-1}xc$.*
(ii) *there exists $z \in G$ such that*
  (a) *$y = z^{-1}xz$, and*
  (b_i) *$g_i = z^{-1}g_iz$ for all $1 \leq i \leq n$.*

Corollary 2.16. *Let A_S be an Artin-Tits group of type B_k ($k \geq 2$) or D_k ($k \geq 4$). Let $X \subseteq S$ be such that Γ_X is connected. In case Γ_S is of type D_{2k+1}, assume that $\{s_2, s_{2'}, s_3\}$ is not included in X with the notations of Figure 5. For any pair (x, y) of elements of A_S, the following are equivalents:*

(i) *there exists $c \in A_X$ such that $y = c^{-1}xc$.*
(ii) *there exists $z \in A_S$ such that*
  (a) *$y = z^{-1}xz$,*
  (b) *$g = z^{-1}gz$ for all g in $\Xi(X)$.*

Proof. By Theorem 0.1 and Proposition 2.1 we have $DZ_G(A_X)) = Z(A_S) \times A_X$. So we are in position to apply Theorem 2.15. $\square$

3. The non spherical type cases

We turn now to the proof of Theorem 0.4 that is concerned with Artin-Tits groups that are not of spherical type. Our main argument is Proposition 3.3. In [20], the second author stated several conjectures, that are proved to hold for Artin-Tits groups of various types (see [18, 19, 20]) as a consequence of a stronger property called Property ($\circledast$) in [20]. This is the case for Artin-Tits groups of spherical type, of FC type, of large type or of 2-dimensional type. then, A_S satisfies Property ($\circledast$) stated in [20]. Property ($\circledast$) is too involved to be explicitly included here. So, we only mention its relevant consequences that will be used in the proof of Proposition 3.3 and refer to [20] for more details.

Proposition 3.1. [20, Proposition 4.3] *Let A_S be an Artin-Tits group. Assume that A_S satisfies Property ($\circledast$). Then $N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X)$ and $QZ_{A_S}(A_X)$ is the subgroup of A_S generated by the set of positive (X, X)-ribbons.*

Remark 3.2. *Let A_S be an Artin-Tits group and X be included in S. An elementary positive ribbons $d_{X,t}$ is defined (see Definition 2.3) only when $X(t)$ is of spherical type. Therefore,*

(i) *when A_X is irreducible and not of spherical type, this is the case if and only if t belongs to $X^\perp$. In this latter case, $d_{X,t}$ is equal to t and is a (X, X)-ribbon. So, for an irreducible and not of spherical type parabolic subgroup A_X, the set of elementary positive $(\cdot, X)$-ribbons is equal to $X^\perp$;*

(ii) *when A_X is of spherical type and not contained in any proper spherical type parabolic subgroup then $d_{X,t}$ is defined only for t in X and in this case $d_{X,t}$ is equal to $\Delta_{X(t)}$ (see Definition 2.3).*

Proposition 3.3. *Let A_S be an Artin-Tits group. Assume that A_S satisfies Property ($\circledast$). Then for any X included in S one has*

(i) *if A_X is of spherical type, then for any positive integer k,*

$$Z_{A_S}(\Delta_X^{2k}) = N_{A_S}(A_X);$$

(ii) *if A_X is of spherical type and there is no A_Y of spherical type with Y strictly containing X, then*

$$QZ_{A_S}(A_X) = QZ(A_X) \text{ and } N_{A_S}(A_X) = A_X;$$

(iii) *if A_X is irreducible and not of spherical type, then*

$$QZ_{A_S}(A_X) = A_{X^\perp} \text{ and } N_{A_S}(A_X) = A_{X\cup X^\perp}.$$

Proof. Assume A_X is irreducible and not of spherical type. By Proposition 3.1, $N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X)$ and $QZ_{A_S}(A_X)$ is generated by the set of positive ribbons, which is, in turn, generated by the set of positive elementary ribbons. By Remark 3.2, this latter set is $X^\perp$. So, $QZ_{A_S}(A_X) = A_{X^\perp}$ and Point (iii) holds. Assume now that A_X is of spherical type. Fix a positive integer k. If g lies in $Z_{A_S}(\Delta_X^{2k})$, then in particular $g^{-1}\Delta_X^{2k}g$ belongs to A_X. Property ($\circledast$) says that, in this case, g belongs to $A_X \cdot QZ_{A_S}(A_X)$, that is to $N_{A_S}(A_X)$ (see Implication (2) $\Rightarrow$ (3) of the definition of Property ($\circledast$) [20, Definition 4.1]). Conversely, $A_X \cdot QZ_{A_S}(A_X)$ is included in $Z_{A_S}(\Delta_X^{2k})$ because both A_X and $QZ_{A_S}(A_X)$ have to fix the center of A_X, which contains Δ_X^2. So Point (i) holds. Finally, if there is no A_Y of spherical type with $X \subsetneq Y$, then the elementary positive ribbons $d_{X,t}$ are the elements $\Delta_{X(t)}$ with t in X (see Remark 3.2 and Definition 2.3). It follows that $QZ_{A_S}(A_X)$ is included in A_X and is, therefore, equal to $QZ(A_X)$. Since $N_{A_S}(A_X) = A_X \cdot QZ_{A_S}(A_X)$, we deduce that $N_{A_S}(A_X) = A_X$. Hence Point (ii) holds. □

In the sequel we first extend Conjecture 0.2 to the context of non irreducible parabolic subgroups (see Conjecture 3.4). Then we prove that Conjecture 3.4 holds for any Artin-Tits group which satisfies Property ($\circledast$) (see Theorem 3.6). Considering Proposition 3.3 (ii), this will prove Theorem 0.4.

Conjecture 3.4. Let A_S be an irreducible Artin-Tits group and X be included in S. Let X_s be the union of the irreducible components of A_X that are of spherical type, and X_{as} be the union of the other irreducible components of X. Then,

$$DZ_{A_S}(A_X) = Z_{A_S}(Z_{A_{X_{as}^\perp}}(A_{X_s}))$$

(i) Assume that X_s is empty. Then

$$DZ_{A_S}(A_X) = Z_{A_S}(A_{X^\perp}).$$

(ii) Assume that A_X is of spherical type. Let A_T be the smallest standard parabolic subgroup of A_S that contains $Z_{A_S}(A_X)$.

(a) If A_T is of spherical type then

$$DZ_{A_S}(A_X) = DZ_{A_T}(A_X).$$

(b) If A_T is not of spherical type then
$$DZ_{A_S}(A_X) = A_X.$$

Proposition 3.5. *Let A_S be an irreducible Artin-Tits group and X be included in S. Assume that A_S satisfies Property ($\circledast$). Conjecture 3.4 implies Conjecture 0.2.*

Proof. Consider the notations of Conjecture 0.2. Assume that X is irreducible. If A_X is not of spherical type, then $X = X_{as}$ and X_s is empty. By Proposition 3.3, $Z_{A_S}(A_X) \subseteq QZ_{A_S}(A_X) = A_{X^\perp} \subseteq Z_{A_S}(A_X)$. Therefore $A_{X^\perp} = Z_{A_S}(A_X)$ and $T = X^\perp$. Thus, Conjecture 3.4(i) implies Conjecture 0.2(i). In the case A_X is of spherical type, there is nothing to prove. □

Theorem 3.6. *Let A_S be an irreducible Artin-Tits group. If A_S satisfies Property ($\circledast$) stated in* [20]*, then Conjecture 3.4 holds.*

In order to prove Theorem 3.6, we need some preliminary results. In the sequel, we assume that A_S is an irreducible Artin-Tits group that satisfies Property ($\circledast$) stated in [20]. We fix a standard parabolic subgroup A_X with $X \subseteq S$. By X_s we denote the union of the irreducible components of A_X that are of spherical type. By X_{as} we denote the union of the other irreducible components of X. By definition, X_s is included in $X_{as}^\perp$. We set
$$\Upsilon = \{Y \subseteq S \mid X_s \subseteq Y;\text{ and } A_Y \text{ is of spherical type.}\}$$
By A_T we denote the smallest standard parabolic subgroup of A_S that contains $Z_{A_S}(A_{X_s})$.

Lemma 3.7. $Z_{A_S}(A_X) = Z_{A_{X_{as}^\perp}}(A_{X_s})$.

Proof. Let $X_1, \cdots, X_k$ be the distinct irreducible components of X_{as}. Then we have $X_{as}^\perp = X_1^\perp \cap \cdots \cap X_k^\perp$. On the other hand, we have $A_{X_{as}} = A_{X_1} \times \cdots \times A_{X_k}$ and $Z_{A_S}(A_{X_{as}}) = Z_{A_S}(A_{X_1}) \cap \cdots \cap Z_{A_S}(A_{X_k})$. By Proposition 3.3, the equality $Z_{A_S}(A_{X_i}) = QZ_{A_S}(A_{X_i}) = A_{X_i^\perp}$ holds for each component X_i. Therefore $Z_{A_S}(A_{X_{as}}) = A_{X_1^\perp} \cap \cdots \cap A_{X_k^\perp} = A_{X_1^\perp \cap \cdots \cap X_k^\perp} = A_{X_{as}^\perp}$. But $A_X = A_{X_s} \times A_{X_{as}}$ and A_{X_s} is included in $A_{X_{as}^\perp}$. Thus, $Z_{A_S}(A_X) = Z_{A_S}(A_{X_s}) \cap Z_{A_S}(A_{X_{as}}) = Z_{A_{X_{as}^\perp}}(A_{X_s})$. □

Lemma 3.8. *The set Υ is not empty and all its elements are contained in T. Moreover, T belongs to Υ if and only if A_T is of spherical type. In this case, T is the unique maximal element of Υ.*

Proof. X_s is contained in Υ, so the latter is not empty. Moreover, X_s is included in T. Therefore the latter belong to Υ if and only if A_T is of spherical type. Finally if Y belongs to Υ, then Δ_Y^2 belongs to $Z_{A_S}(A_Y)$, and therefore to $Z_{A_S}(A_X)$. Thus, Y is included in T. Hence, if T belongs to Υ, it is its unique maximal element. □

Lemma 3.9. *Assume that Y is maximal in Υ for the inclusion. Then,*
$$DZ_{A_S}(A_{X_s}) \subseteq DZ_{A_Y}(A_{X_s})$$

Proof. Assume that g belongs to $DZ_{A_S}(A_{X_s})$. The element Δ_Y^2 lies in $Z(A_Y)$. Since X_s is included in Y, it follows that Δ_Y^2 lies in $Z_{A_S}(A_{X_s})$, and $g\Delta_Y^2 g^{-1} = \Delta_Y^2$. By Proposition 3.3(i)(a), g belongs to the subgroup $N_{A_S}(A_Y)$. But Y is maximal in Υ. By Proposition 3.3(i)(b), $N_{A_S}(A_Y) = A_Y$. Thus $DZ_{A_S}(A_{X_s}) = Z_{A_S}(Z_{A_S}(A_{X_s})) \cap A_Y = Z_{A_Y}(Z_{A_S}(A_{X_s})) \subseteq Z_{A_Y}(Z_{A_Y}(A_{X_s})) = DZ_{A_Y}(A_{X_s})$. □

We can now prove Theorem 3.6

Proof of Theorem 3.6. By Lemma 3.7, we have $Z_{A_S}(A_X) = Z_{A_{X_{as}^\perp}}(A_{X_s})$. It follows that $DZ_{A_S}(A_X) = Z_{A_S}(Z_{A_{X_{as}^\perp}}(A_{X_s}))$. When X_s is empty, we have $X_{as} = X$ and $A_{X_s} = \{1\}$. So $Z_{A_{X_{as}^\perp}}(A_{X_s}) = Z_{A_{X^\perp}}(\{1\}) = A_{X^\perp}$. Therefore, $DZ_{A_S}(A_X) = Z_{A_S}(A_{X^\perp})$. Assume that for the remaining of the proof that A_X is of spherical type. Assume, first, that A_T is of spherical type. By Lemma 3.8, T is maximal in Υ and, by Lemma 3.9, $DZ_{A_S}(A_X) \subseteq DZ_{A_T}(A_X)$. On the other hand, by the definition of T we have $Z_{A_S}(A_X) \subseteq A_T$. Therefore, $Z_{A_T}(A_X) = Z_{A_S}(A_X) \cap A_T = Z_{A_S}(A_X)$. We deduce that $DZ_{A_T}(A_X) = Z_{A_T}(Z_{A_S}(A_X)) \subseteq DZ_{A_S}(A_X)$. Hence, $DZ_{A_S}(A_X) = DZ_{A_T}(A_X)$. Assume, finally, that T does not lie in Υ. Let Y be maximal in Υ. By Lemma 3.9, we get $DZ_{A_S}(A_X) \subseteq DZ_{A_Y}(A_X)$. If $Y = X$, then $A_X \subseteq DZ_{A_S}(A_X) \subseteq DZ_{A_X}(A_X) = A_X$ and we are done. So, assume that $X \subsetneq Y$. The group A_Y is of spherical type. Applying Theorem 0.1, we get that $DZ_{A_Y}(A_X) \subseteq QZ(A_Y) \ltimes A_X$. Since A_X is included in $DZ_{A_S}(A_X)$, the group A_X is equal to $DZ_{A_S}(A_X)$ if and only if $DZ_{A_S}(A_X) \cap QZ(A_Y) = \{1\}$. Assume this is not the case. Then, there exists $k > 0$ so that Δ_Y^k lies in $DZ_{A_S}(A_X)$. We can assume without restriction that k is even. Since Y lies in Υ and T does not, they are distinct. It follows from the definition of T that there exists g in $Z_{A_S}(A_X)$ which is not in A_Y. But Δ_Y^k lies in $DZ_{A_S}(A_X)$. So we have $\Delta_Y^k g(\Delta_Y^k)^{-1} = g$, and equivalently $g\Delta_Y^k g^{-1} = \Delta_Y^k$. The latter equality imposes that g belongs to $N_{A_S}(A_Y)$ by Proposition 3.3(i)(a). But $N_{A_S}(A_Y) = A_Y$ by Proposition 3.3(i)(b), a contradiction. Hence, $DZ_{A_S}(A_X) = A_X$. □

Corollary 3.10. *Let A_S be an irreducible Artin-Tits group of FC type, or of large type, or of 2-dimensional type. Then, Conjecture 0.2 holds.*

Proof. Under the assumption, Conjecture 3.4 holds and implies Conjecture 0.2. □

Remark 3.11. *In an (irreducible) Artin-Tits group that is of large type, all standard parabolic subgroups are irreducible. So, Corollary 3.10 provides a complete description of the double centralizer of any standard parabolic subgroups. However, for the other non-spherical types in the case both X_s and X_{as} are not empty, the answer is not completely satisfactory. Indeed the double centralizer is not as simple as in the cases where either X_s or X_{as} is empty. For instance, in the two following examples, the reader may verify that $Z_{A_S}(A_X) = Z(A_{X_s})$ and $DZ_{A_S}(A_X) = N_{A_S}(A_{X_s}) = QZ_{A_S}(A_{X_s}) \cdot A_{X_s}$*

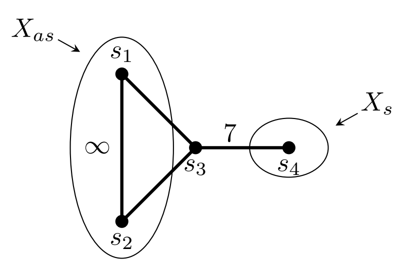


FIGURE 8. An Artin-Tits groups of 2-dimensional type.

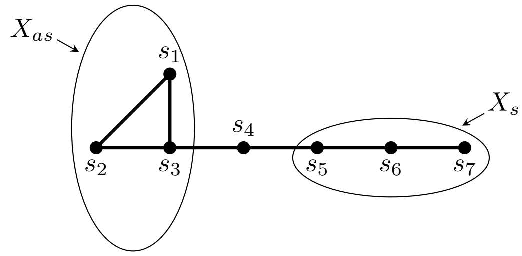


FIGURE 9. An Artin-Tits groups of small type ($m_{s,t} \leq 3$ for all s, t).

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