



**HAL**  
open science

# Effective Wavenumber of a Random Polydisperse Assembly of Poroelastic Spheres

Hervé Franklin

► **To cite this version:**

Hervé Franklin. Effective Wavenumber of a Random Polydisperse Assembly of Poroelastic Spheres. Romanian Journal of acoustics and vibration, 2019. hal-02128513

**HAL Id: hal-02128513**

**<https://hal-normandie-univ.archives-ouvertes.fr/hal-02128513>**

Submitted on 14 May 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

---

---

# Effective Wavenumber of a Random Polydisperse Assembly of Poroelastic Spheres

**Hervé FRANKLIN**

*Laboratoire Ondes et Milieux Complexes (LOMC) UMR CNRS 6294 Université Le Havre Normandie, France, herve.franklin@univ-lehavre.fr*

*Abstract* - The effective wavenumber model of Linton and Martin [1] for the dilute monodisperse case (obstacles of identical sizes) is used to calculate the effective modulus and mass density of a polydisperse assembly of poroelastic spheres. We focus on the Rayleigh limit (low frequency regime) where the wavelengths can be considered very large compared to the size of the obstacles. In order to show the influence of the distribution in size of the obstacles, the special case of Schulz probability density function is considered.

*Keywords:* - Random media, effective moduli, polydispersity.

---

## 1. INTRODUCTION

The properties of the coherent wave propagating in a fluid containing randomly distributed poroelastic spheres are studied. The polydispersity - defined as the variation in size of the obstacles in a representative volume - is taken into account. The starting point is the effective wavenumber model of Linton and Martin (LM) [1] in the monodisperse case (identical sizes). This model based on a closure assumption often called the Quasi-Crystalline Approximation (QCA) of Lax, is an improvement of the models of Foldy [2] and Waterman and Truell [3] initially which were built for the dilute case (low concentrations). The following effective quantities will be derived: wavenumber, modulus and mass density of a polydisperse suspension in the low frequency regime where the wavelength can be assumed very large compared to the size of the obstacles (or large with regard to the representative volume). The monodisperse case (all the obstacles have the same radius) can then be seen as a particular case of polydispersity.

The phenomenon of polydispersity is met in dispersions [4] and colloids [5-6]. It also manifests in the flowing and transport of solid particles [7] and in the distribution of air bubbles in fluids [8-10]. An understanding of the effects induced by the distribution in size of the obstacles on the effective acoustic properties of the dispersions can be of great interest, especially to higher orders in concentration - it should be noted that so far, the most used formulas are those of Foldy which only accounts first order terms in concentration and Waterman and Truell which contains an incorrect second order term. Moreover, considering porous obstacles, previous results on elastic spheres are extended.

## 2. BACKGROUND

The fluid surrounding the porous obstacles has a mass density  $\rho_0$  and a speed of sound  $c_0$ . The fluid saturating the porous space has a mass density  $\rho_f$ , a speed of sound  $c_f$  and a kinematic viscosity  $\eta$ . It will be assumed for the sake of simplicity that  $\rho_f = \rho_0$  and  $c_f = c_0$ . We denote by  $\rho_s$  the mass density of the material constituting the solid grains and by  $\rho = (1 - \phi)\rho_s + \phi\rho_0$  the density of the fluid saturated porous medium of porosity  $\phi$ . The porous medium obeys Biot's theory and is also characterized by a tortuosity  $\tilde{\alpha}$  sometimes referred to as the structure factor [11] (see also Table 1, Ref. [12] for a complete list of the parameters constituting the saturated porous medium).

Let  $\omega$  be the angular frequency and  $k_0 = \omega/c_0$  the wavenumber of the longitudinal incident wave in the absence of the obstacles. When it encounters the obstacles, the incident wave is partially reflected. The part of the wave penetrating the obstacles is converted into three waves according to Biot's theory of poroelastic media : two longitudinal waves of respective wavenumbers  $k_1 = \omega/c_1$  (fast) and  $k_2 = \omega/c_2$  (slow) and a transverse wave of wavenumber  $k_t = \omega/c_t$ . Since the three wavenumbers are all complex-valued, the waves attenuate when propagating in the porous space. The scattering of an incident plane wave by a spherical obstacle of radius  $a$  is described by the scattering coefficients  $T_n$  ( $T_{-n} = T_n$ ) where  $n$  is a relative integer (*cf.* Appendix A). The  $T_n$ s depend on spherical Bessel and Hankel functions which in

tour depend on one of the normalized wavenumbers  $k_\alpha a$  ( $\alpha = 0, 1, 2, t$ ).

The far-field scattering function  $f(\theta)$  is a modal series depending on the angle of observation  $\theta$  and on the coefficients  $T_n$  as follows

$$f(\theta) = \frac{1}{ik_0} \sum_{n=0}^{+\infty} (2n+1) T_n P_n(\cos\theta). \quad (1)$$

It is used in the LM model discussed later. The Legendre polynomial  $P_n(\cos\theta)$  of order  $n$  is equal to 1 for  $\theta=0$  (forward scattering) or  $\theta=\pi$  (backscattering). At low frequencies the normalized wavenumbers  $k_\alpha a$  are such that  $|k_\alpha a| \ll 1$  ( $\alpha = 0, 1, 2, t$ ). Taylor series expansions for these functions show that

$$T_0(a) = \frac{i}{3} (k_0 a)^3 (B_0 - 1) + O((k_0 a)^5) \quad (2)$$

$$T_1(a) = \frac{i}{3} (k_0 a)^3 B_1 + O((k_0 a)^5) \quad (3)$$

$$T_n(a) = O((k_0 a)^5) \quad \text{if } n \geq 2 \quad (4)$$

The coefficients  $B_0$  and  $B_1$  are given in Appendix B. The expansions Eqs. (2-4) for a single obstacle are usually called "Rayleigh limits". The study for any assembly of obstacles presented here remaining restricted to low frequencies, we keep this denomination below.

### 3. LINTON AND MARTIN EFFECTIVE WAVENUMBER. POLYDISPERSITY.

In 1967, using calculations based on methods of nuclear physics, Lloyd and Berry [13] extended the model of the effective wavenumber of Waterman and Truell for the case of spherical obstacles. They added to the second order in concentration of this wavenumber an integral formula which accounts for a summation on all the scattering angles. Four decades later, working on a new proof of the results of Lloyd and Berry from a more classical approach, LM [1] proposed in the case of cylindrical obstacles a rewrite of the effective wavenumber equivalent to that of Lloyd and Berry, the integral on the angles being transformed into a double series whose general term is proportional to the product  $T_n T_m$  of the scattering coefficients. Although extended to any

order in  $n_0$  for cylinders [14] and for spheres [15], we consider here only the initial formula of LM which stops at the second order in  $n_0$ .

Let us start from a fluid medium containing  $n_0$  obstacles per unit volume. The distribution of radius  $a$  is assumed to be continuous and extends from  $a_1 = 0$  to  $a_2 = +\infty$ . It is characterized by a probability density function  $\eta$  such that  $\int_0^{+\infty} \eta(a) da = 1$ . The effective wavenumber of LM is given by [1]

$$\xi_{LM}^2 = k_0^2 + n_0 \delta_1 + n_0^2 \delta_2 + O(n_0^3) \quad (5)$$

where

$$\delta_1 = \frac{4\pi}{ik_0} \int_{a_1}^{a_2} \sum_{n=0}^{+\infty} (2n+1) T_n(a) \eta(a) da \quad (6)$$

and

$$\delta_2 = -\frac{1}{2} \left( \frac{4\pi}{k_0} \right)^4 \times \int_{a_1}^{a_2} \int_{a_1}^{a_2} \sum_{n,m=0}^{+\infty} K_{nm} T_n(a) T_m(b) \eta(a) \eta(b) da db \quad (7)$$

In Eq. (7)

$$K_{nm} = \left( \frac{1}{4\pi} \right)^{3/2} \sqrt{(2n+1)(2m+1)} \times \sum_{q=|n-m|}^{n+m} q \sqrt{2q+1} G(n, 0; m, 0; q) \quad (8)$$

where  $G$  is the Gaunt coefficient. The sum on the index  $q$  is in steps of two with  $n+m+q$  even. Algorithms are provided for the fast calculation of Gaunt coefficients [16].

When the frequency domain of study is arbitrary, one must evaluate numerically the single and double integrals appearing in Eqs. (6-7). The low-frequency approximations of the scattering coefficients  $T_n$  given above make it possible to simplify considerably the computations of the integrals. Considering Eqs. (2-4) and (5-7) we obtain

$$\xi_{LM}^2 \approx k_0^2 (1 + I_1 + I_2) \quad (9)$$

where

$$I_1 = (B_0 - 1 + 3B_1) n_0 \frac{4\pi}{3} \langle a^3 \rangle \quad (10)$$

and

$$I_2 = 3 \left[ (B_0 - 1) B_1 + 2B_1^2 \right] \left[ n_0 \frac{4\pi}{3} \langle a^3 \rangle \right]^2 \quad (11)$$

Above, we have introduced the notation

$$\langle a^n \rangle = \int_0^{+\infty} a^n \eta(a) da \quad (12)$$

which represents the moment of order  $n$  of the probability density function (if  $n=1$  it is the average radius  $\langle a \rangle$  of the obstacles). Note that  $K_{01} = K_{10} = 3/(16\pi^2)$  and  $K_{11} = 3/(4\pi^2)$  are also used. Equations (10-11) show that there are only two integrals to compute,  $\langle a^2 \rangle$  and  $\langle a^3 \rangle$ . In the following  $C_V = n_0(4\pi/3)\langle a^3 \rangle$  will denote the average concentration. It is clear that the concentration does not depend on the physical parameters of the media.

### 3.1. Concentrations

We will examine the effect of polydispersity in the case of the Schulz distribution, Fig. 1, and then deduce the formula of the concentration. In their work, Leroy et al. [9] considered the log-normal distribution, Mascaro *et al.* [10] the Gaussian distribution which can be seen as a particular case of the Schulz distribution. The Schulz distribution [18-21] is a special case of the Gamma distribution. Its probability density function is given by

$$\eta(a) = \frac{(j+1)^{j+1}}{\langle a \rangle \Gamma(j+1)} \left( \frac{a}{\langle a \rangle} \right)^j \exp \left[ -(j+1) \frac{a}{\langle a \rangle} \right] \quad (13)$$

where  $\Gamma(\cdot)$  is Euler's gamma function. The number  $j$  measures the width of the distribution and is connected to the polydispersity because  $p = \sigma/\langle a \rangle = 1/\sqrt{j+1}$  ( $\sigma^2$  is the variance). The case  $j=0$  gives  $p=1$  and corresponds to the exponential distribution (where the number of small-radius diffusers is preponderant).

When  $j$  is large,  $p$  is small; the dispersion is small and the distribution is close to a Gaussian. In the following,  $j$  is a positive integer and we use the relation  $\Gamma(j+1) = j!$ . The moment of order  $n$  is then given by

$$\langle a^n \rangle = \int_0^{+\infty} a^n \eta(a) da = \frac{(j+n)!}{j!} \frac{\langle a \rangle^n}{(j+1)^n} \quad (14)$$

The average concentration for a distribution of spherical obstacles is

$$\begin{aligned} C_V &= n_0 \frac{4\pi \langle a \rangle^3}{3} \frac{(j+3)(j+2)}{(j+1)^2} \\ &= n_0 \langle \mathcal{V} \rangle (1+3p^2+2p^4) \end{aligned} \quad (15)$$

where  $\langle \mathcal{V} \rangle = 4\pi \langle a \rangle^3/3$ . The special case of monodispersity is obtained when  $j \rightarrow +\infty$  or  $p \rightarrow 0$  (the probability density function then tends towards an increasingly narrow peak of increasing amplitude and centered on the value 1). The result is the following limits  $\langle a^n \rangle \rightarrow \langle a \rangle^n = a^n$  ( $n \geq 1$ ),  $\sigma \rightarrow 0$  and  $C_V \rightarrow n_0 V$ .

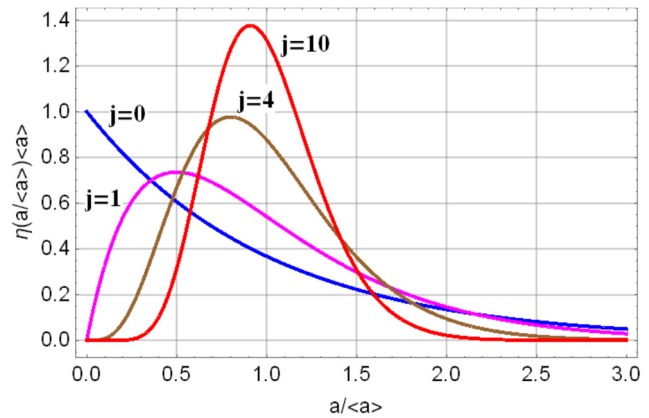


Figure 1. The Schulz distribution

### 3.2. Effective wavenumber

Making the appropriate substitutions in Eq. (9), one arrives at a formula for the effective wavenumber going up to the second order in concentration of the form

$$\begin{aligned} \left( \frac{\xi_{LM}}{k_0} \right)^2 &= 1 + C_V [B_0 - 1 + 3B_1] + \\ &3C_V^2 [B_1(B_0 - 1) + 2B_1^2] \end{aligned} \quad (16)$$

As long as terms of order  $C_V^3$  and higher can be neglected, Eqs. (16) can be rewritten as [12, 22]

$$\left( \frac{\xi_{LM}}{k_0} \right)^2 = [1 + C_V(B_0 - 1)] [1 + 3C_V B_1 + 6C_V^2 B_1^2] \quad (17)$$

## 4. EFFECTIVE MODULUS

### 4.1. Porous obstacles

In the absence of obstacles, the square of the wavenumber in the fluid medium is given by  $k_0^2 = \omega^2/c_0^2$ , where  $c_0^2 = K_0/\rho_0$  ( $K_0$  is the bulk modulus). We define, by analogy with this last formula, the square of the effective wavenumber  $\xi_{LM}^2 = \omega^2(\rho_{LM}/M_{LM})$  of the medium containing the obstacles, where  $\rho_{LM}$  represents the effective mass density and  $M_{LM}$  the effective modulus. The ratio of the squared wavenumbers gives [22]:

$$\left(\frac{\xi_{LM}}{k_0}\right)^2 = \frac{\rho_{LM}}{\rho_0} \times \frac{K_0}{M_{LM}} \quad (18)$$

Comparing with Eq. (17), we deduce that

$$\frac{\rho_{LM}}{\rho_0} = 1 + 3C_V B_1 + 6C_V^2 B_1^2 \quad (19)$$

and that

$$\frac{1}{M_{LM}} = \frac{1-C_V}{K_0} + \frac{C_V B_0}{K_0} \quad (20)$$

Accounting for the explicit form of  $C_V$ , Eq. (15), we have also

$$\frac{\rho_{LM}}{\rho_0} = 1 + 3n_0 \langle \mathcal{V} \rangle (1 + 3p^2 + 2p^4) B_1 + 6n_0^2 \langle \mathcal{V} \rangle^2 (1 + 6p^2 + 13p^4 + 12p^6 + 4p^8) B_1^2 \quad (19')$$

and

$$\frac{1}{M_{LM}} = \frac{1 - n_0 \langle \mathcal{V} \rangle (1 + 3p^2 + 2p^4)}{K_0} + \frac{n_0 \langle \mathcal{V} \rangle (1 + 3p^2 + 2p^4) B_0}{K_0} \quad (20')$$

from which the monodisperse case can be derived by putting  $p = 0$ . It can be seen that the ratio  $\rho_{LM}/\rho_0$  only takes into account  $B_1$  while  $M_{LM}$  only takes into account  $B_0$ . In Appendix B, substituting Eqs. (B4-B7) in Eqs. (B2-B3) we arrive at the static expressions of  $B_0$  and  $B_1$  given in Eqs. (B8-B9). We thus obtain simplified forms of Eqs (19-20) showing

the role played by the parameters characterizing the porous medium

$$\frac{\rho_{LM}}{\rho_0} = 1 + 3C_V \frac{(1-\phi)(\rho_s - \rho_0)}{2(1-\phi)\rho_s + (1+2\phi)\rho_0} + \dots \quad (21)$$

$$\frac{1}{M_{LM}} = \frac{1-C_V}{K_0} + \frac{C_V}{K_0} \frac{A}{\bar{A}} \quad (22)$$

where

$$A = 4 - \left(1 - \frac{H}{C}\right) \left[4 - 3 \frac{H}{\mu} \left(1 - \frac{\rho_0}{\rho}\right)\right] - 3 \frac{HM - C^2}{C\mu}$$

and

$$\bar{A} = \left(4 - 3 \frac{H}{\mu}\right) \frac{HM - C^2}{CK_0} + 4 \frac{\rho_0 H}{\rho K_0}$$

The interpretation of the bulk modulus, Eq. (22), is the following: the trace of the mean deformation is given by  $Tr\langle \varepsilon \rangle = -P/M_{LM}$  if  $P$  represents the hydrostatic pressure. Equations (31-32) do not depend on the viscosity of the fluid saturating the pores nor on the permeability. However, they still depend on the porosity.

### 4.2. Elastic obstacles

When the porosity tends towards zero ( $\phi \rightarrow 0$ ), the medium becomes an elastic solid described by the Lamé constants  $\lambda_s$ ,  $\mu_s$  and the bulk modulus  $K_s = \lambda_s + 2\mu_s/3$ . We have

$$H \rightarrow H_s = K_s + \frac{4\mu_s}{3}, \quad \mu \rightarrow \mu_s, \quad C \rightarrow C_s = K_s \quad (23)$$

and then

$$B_0 \rightarrow E_0 = \frac{K_0}{\lambda_s + 2\mu_s/3} = \frac{3\rho_0 c_0^2}{\rho_s (3c_L^2 - 4c_T^2)} \quad (24)$$

$$B_1 \rightarrow E_1 = \frac{\rho_s - \rho_0}{2\rho_s + \rho_0} \quad (25)$$

In Eqs. (24), we used the relations  $\lambda_s = \rho_s (c_L^2 - 2c_T^2)$  and  $\mu_s = \rho_s c_T^2$ , where  $c_L$  ( $c_T$  respectively) represents the phase velocity of the longitudinal wave (transverse wave respectively) in the elastic medium. Substitutions of  $B_0$  by  $E_0$  and  $B_1$  by  $E_1$  in Eqs. (19-20) give the static expressions

$$\frac{\rho_{LM}}{\rho_0} \xrightarrow{\phi \rightarrow 0} 1 + 3C_V \frac{\rho_s - \rho_0}{2\rho_s + \rho_0} + 6C_V^2 \left( \frac{\rho_s - \rho_0}{2\rho_s + \rho_0} \right)^2 \quad (26)$$

and

$$\frac{1}{M_{LM}} \xrightarrow{\phi \rightarrow 0} \frac{1 - C_V}{K_0} + \frac{C_V}{\lambda_s + 2\mu_s/3} \quad (27)$$

Equation (26) provides an extension to order 2 in concentration of the results given to order 1 by Aristégui and Angel who used the wavenumber of Waterman and Truell and considered a monodisperse distribution of obstacles (see Eqs. (37) and (41) of Ref. [23]). In the dilute case and to the first order in concentration, the results of Kuster and Toksöz [24] agree with those given here. The divergence observed from the second order in concentration between the effective mass densities of LM and of Kuster and Toksoz comes from the fact that the latter do not consider the multiple scattering phenomenon between the spheres.

## 5. CONCLUSIONS

In this paper, we have used the general formula of the effective wavenumber provided by LM to determine the properties - mass densities and effective moduli - of polydisperse media made up of poroelastic spherical obstacles immersed in a fluid. Schulz's statistical distribution was considered for the size of the obstacles. Formulas are obtained for the effective wavenumber, modulus and mass density at low frequency. From there we derive formulas in the static case for porous obstacles and next for elastic obstacles. All formulas relative to the effective quantities depend on the fundamental parameter of concentration, which in turn depends closely on the statistical distribution considered for the size of the obstacles.

## APPENDIX A

For a poro-elastic sphere surrounded and saturated by the same liquid, the application of the open-pore boundary conditions (Eq. (A1) of Ref. [12]) allows to write the scattering coefficients  $T_n$  as a ratio of determinants of order 4  $d_n$  and  $d_n^{[1]}$ :

$$T_n = \frac{d_n^{[1]}}{d_n}. \quad (A1)$$

The elements of  $d_n$  are

$$d_{11} = x_0 h_n^{(1)'}(x_0), \quad d_{12} = -\Gamma_1 x_1 j_n'(x_1),$$

$$\begin{aligned} d_{13} &= -\Gamma_2 x_2 j_n'(x_2), \\ d_{14} &= -n(n+1)\Gamma_t j_n(x_t), \end{aligned} \quad (A2)$$

$$\begin{aligned} d_{21} &= -h_n^{(1)}(x_0), \quad d_{22} = \rho_{f10} j_n(x_1), \\ d_{23} &= \rho_{f20} j_n(x_2), \quad d_{24} = 0, \end{aligned} \quad (A3)$$

$$\begin{aligned} d_{31} &= -\rho_{0t} x_t^2 h_n^{(1)}(x_0), \\ d_{32} &= 4x_1 j_n'(x_1) + (\rho_{1t} x_t^2 - 2n(n+1)) j_n(x_1), \\ d_{33} &= 4x_2 j_n'(x_2) + (\rho_{2t} x_t^2 - 2n(n+1)) j_n(x_2) \\ d_{34} &= -2n(n+1) [x_t j_n'(x_t) - j_n(x_t)] \end{aligned} \quad (A4)$$

$$\begin{aligned} d_{41} &= 0, \quad d_{42} = x_1 j_n'(x_1) - j_n(x_1), \\ d_{43} &= x_2 j_n'(x_2) - j_n(x_2), \\ d_{44} &= -x_t j_n'(x_t) + (1 - n(n+1) + x_t^2/2) j_n(x_t) \end{aligned} \quad (A5)$$

The determinant  $d_n^{[1]}$  is derived from  $d_n$  by replacing the elements of the first column of  $d_n$  with the following

$$\begin{aligned} \alpha_{11} &= -x_0 j_n'(x_0), & \alpha_{21} &= j_n(x_0), \\ \alpha_{31} &= \rho_{0t} x_t^2 j_n(x_0); & \alpha_{41} &= 0. \end{aligned} \quad (A6)$$

Above,  $\Gamma_\alpha = 1 + \gamma_\alpha$  ( $\alpha = 1, 2, t$ ) are compatibility coefficients [25],  $x_\alpha = k_\alpha a = \omega a / c_\alpha$  ( $\alpha = 0, 1, 2, t$ ) are dimensionless wavenumbers and  $j_n, h_n^{(1)}$  are the spherical Bessel and Hankel functions of order  $n$  [26]. Further,

$$\begin{aligned} \gamma_\alpha &= \frac{Hk_\alpha^2 - \rho\omega^2}{\rho_f \omega^2 - Ck_\alpha^2} \quad (\alpha = 1, 2) \\ \gamma_t &= \frac{\mu k_t^2 - \rho\omega^2}{\rho_f \omega^2} \end{aligned} \quad (A7)$$

denote dimensionless coefficients. The elastic moduli  $H, \mu$  and  $C$  (to which a fourth module  $M$  must be associated) characterize the porous medium [25]. The coefficients  $\rho_{\alpha t} = \rho_\alpha / \rho_t$  ( $\alpha = 0, 1, 2$ ) and  $\rho_{f\alpha 0} = \rho_{f\alpha} / \rho_0$ ,  $\rho_{f\alpha 0} = \rho_{f\alpha} / \rho_0$  ( $\alpha = 1, 2$ ) represent mass density ratios [27] where

$$\begin{aligned} \rho_\alpha &= \rho_0 \gamma_\alpha + \rho \quad (\alpha = 1, 2, t), \\ \rho_{f\alpha} &= \rho_0 \left( 1 + \gamma_\alpha \frac{\tilde{\alpha}}{\phi} \right) \quad (\alpha = 1, 2). \end{aligned} \quad (A8)$$

It is recalled that  $\tilde{\alpha}$  is the dynamic tortuosity [11],  $\rho = (1 - \phi)\rho_s + \phi\rho_0$  the mass density of the porous medium and  $\phi$  the porosity.

## APPENDIX B

The coefficients  $B_0$  and  $B_1$  of Eqs. (1-2) are obtained from Taylor's expansions of the scattering coefficients  $T_n$  for  $|x_\alpha|$  ( $\alpha = 0, 1, 2, t$ ) small compared to one, Appendix A. We have

$$B_0 = \frac{B_{0n1}\Gamma_2 - B_{0n2}\Gamma_1}{B_{0d1} - B_{0d2}} \quad (B1)$$

where

$$B_{0n1} = \left( 4 - 3 \frac{\rho_1 c_1^2}{\rho_t c_t^2} + 3 \frac{\rho_{f1} c_1^2}{\rho_t c_t^2} \right)$$

$$B_{0n2} = \left( 4 - 3 \frac{\rho_2 c_2^2}{\rho_t c_t^2} + 3 \frac{\rho_{f2} c_2^2}{\rho_t c_t^2} \right)$$

$$B_{0d1} = \frac{\rho_{f2} c_2^2}{\rho_0 c_0^2} \left( 4 - 3 \frac{\rho_1 c_1^2}{\rho_t c_t^2} \right)$$

$$B_{0d2} = \frac{\rho_{f1} c_1^2}{\rho_0 c_0^2} \left( 4 - 3 \frac{\rho_2 c_2^2}{\rho_t c_t^2} \right)$$

and

$$B_1 = -\frac{B_{1n1} + B_{1n2} + B_{1n3}}{B_{1d1} + B_{1d2} + B_{1d3}} \quad (B2)$$

$$B_{1n1} = 4 \frac{c_2^2}{c_1^2} (-\Gamma_2 + \rho_{f20} + \Gamma_t \Delta \rho_2)$$

$$B_{1n2} = 4 (\Gamma_1 - \rho_{f10} - \Gamma_t \Delta \rho_1)$$

$$B_{1n3} = 3 \frac{c_2^2}{c_t^2} (\Gamma_2 \Delta \rho_1 - \Gamma_1 \Delta \rho_2 + \Delta \rho_{12})$$

$$B_{1d1} = 4 \frac{c_2^2}{c_1^2} (-\Gamma_2 - 2\rho_{f20} + \Gamma_t \Delta \rho_2)$$

$$B_{1d2} = -4 (-\Gamma_1 - 2\rho_{f10} + \Gamma_t \Delta \rho_1)$$

$$B_{1d3} = 3 \frac{c_2^2}{c_t^2} (\Gamma_2 \Delta \rho_1 - \Gamma_1 \Delta \rho_2 - 2\Delta \rho_{12})$$

where

$$\begin{aligned} \Delta \rho_2 &= \rho_{2t} - \rho_{f2t} \\ \Delta \rho_1 &= \rho_{1t} - \rho_{f1t} \\ \Delta \rho_{12} &= \rho_{2t} \rho_{f10} - \rho_{1t} \rho_{f20} \end{aligned} \quad , \quad (B3)$$

with  $\rho_{f\alpha t} = \rho_{f\alpha} / \rho_t$  ( $\alpha = 1, 2$ ).

The above parameters as well as fast, slow and shear wave speeds all depend on the frequency. The same is true of the coefficients  $B_0$  and  $B_1$ . At low frequencies the following approximations can be used (the porosity of the medium being fixed and different from zero)

$$c_2^2 \approx -i \frac{\omega}{\omega_c} \frac{HM - C^2}{\rho_0 H}, \quad c_1^2 \approx \frac{H}{\rho}, \quad c_t^2 \approx \frac{\mu}{\rho} \quad (B4)$$

$$\gamma_1 = 0, \quad \gamma_2 \approx -\frac{H}{C}, \quad \gamma_t = 0 \quad (B5)$$

$$\rho_1 \approx \rho, \quad \rho_2 \approx \rho - \frac{\rho_0 H}{C}, \quad \rho_t \approx \rho, \quad (B6)$$

$$\rho_{f1} \approx \rho_0, \quad \rho_{f2} \approx -\frac{i\omega_c}{\omega} \frac{H\rho_0}{C}, \quad (B7)$$

where  $\omega_c = \eta / (\rho_0 \kappa)$  is a characteristic frequency built with the permeability  $\kappa$  of the porous medium, the mass density  $\rho_0$  and the kinematic viscosity  $\eta$  of the saturating fluid. We obtain

$$B_0 \approx \frac{B_{0n1} + B_{0n2} + 4}{B_{0d1} + B_{0d2}} \quad (B8)$$

$$B_{0n1} = \frac{3(C^2 - HM)}{C\mu}$$

$$B_{0n2} = -\left[ 3 \frac{\rho_0 H}{\rho\mu} + 4 - 3 \frac{H}{\mu} \right] \left( 1 - \frac{H}{C} \right)$$

$$B_{0d1} = \left( 4 - 3 \frac{H}{\mu} \right) \left( \frac{HM - C^2}{CK_0} \right)$$

$$B_{0d2} = 4 \frac{\rho_0 H}{\rho K_0}$$

and

$$B_1 \approx \frac{\rho - \rho_0}{2\rho + \rho_0}. \quad (B9)$$

and see that coefficients  $B_0$  and  $B_1$  no longer depend on the frequency. We are in the static limit where the only physical parameters playing a role are those of the fluid ( $\rho_0$ ,  $K_0 = \rho_0 c_0^2$ ) and the porous medium ( $H$ ,  $M$ ,  $C$ ,  $\mu$ ). We have

$$H = \frac{(K_s - K_b)^2}{D_b - K_b} + K_b + \frac{4\mu}{3} \quad (B10)$$

$$C = \frac{K_s (K_s - K_b)}{D_b - K_b} \quad (B11)$$

$$M = \frac{K_s^2}{D_b - K_b} \quad M = \frac{K_s^2}{D_b - K_b} \quad (\text{B12})$$

$$D_b = K_s \left[ 1 + \phi \left( \frac{K_s}{K_0} - 1 \right) \right] \quad (\text{B13})$$

where  $K_s$  ( $\mu$ , respectively) is the bulk modulus (shear modulus, respectively) of the frame material and  $K_b$  the bulk modulus of the skeletal frame.

## REFERENCES

- [1] Linton C. M., Martin P. A., Multiple Scattering by Multiple Spheres: A New Proof of the Lloyd-Berry Formula for the Effective Wavenumber, *SIAM Journal on Applied Mathematics* 66 (5), 2006, pp. 1649–1668.
- [2] Foldy L. L., The multiple scattering of waves. General theory of isotropic scattering by randomly distributed scatterers. *Phys. Rev.* 67, 1945, pp. 107–119.
- [3] Waterman P. C., Truell R., Multiple scattering of waves, *J. Math. Phys.* 2 (4), 1961, pp. 512–537.
- [4] Pinfield V. J., Forrester D. M., Multiple scattering in random dispersions of spherical scatterers: Effects of shear-acoustic interactions, *J. Acoust. Soc. Am.* 141 (1), 2017, pp. 649–660.
- [5] Russell W. B., Saville D. A., Schowalter W. R., *Colloidal dispersions*, Cambridge University Press, Cambridge, England, 1989.
- [6] Challis R. E., Povey M. J. W., Mather M. L., Holmes A. K., Ultrasound techniques for characterizing colloidal dispersions, *Rep. Prog. Phys.* 68, 2005, pp. 1541–1637.
- [7] Scheidegger A. E., *The Physics of Flow Through Porous Media*, University of Toronto Press, Toronto, Canada, 1974.
- [8] Gaunaurd G. C., Uberall H., Resonance theory of bubbly liquids, *J. Acoust. Soc. Am.* 69 (2), 1981, pp. 362–370.
- [9] Leroy V., Strybulevych A., Page J. H., Scanlon M. G., Sound velocity and attenuation in bubbly gels measured by transmission experiments, *J. Acoust. Soc. Am.* 123 (4), 2008, pp. 1931–1940.
- [10] Mascaro B., Brunet T., Poncelet O., Aristégui C., Raffy S., Mondain-Monval O., Leng J., Impact of polydispersity on multipolar resonant scattering in emulsions, *J. Acoust. Soc. Am.* 133 (4), 2013, pp. 1996–2003.
- [11] Johnson D. L., Koplik J., Dashen R., Theory of dynamic permeability and tortuosity in fluid-saturated porous media, *J. Fluid Mech.* 176, 1987, pp. 379–402.
- [12] Franklin H., Rayleigh limits for effective wavenumbers of randomly distributed porous cylinders. Comparison of explicit and implicit methods, *Wave Motion* 66, 2016, pp. 106–117.
- [13] Lloyd P., Berry M. V., Wave propagation through an assembly of spheres: IV. Relations between different multiple scattering theories, *Proc. Phys. Soc.* 91 (3), 1967, pp. 678–688.
- [14] Norris A. N., Conoir J.-M., Multiple scattering by cylinders immersed in fluid: high order approximations for the effective wavenumbers, *J. Acoust. Soc. Am.* 129 (1), 2011, pp. 104–113.
- [15] Luppé F., Conoir J.-M., Norris A. N., Effective wave numbers for thermo-viscoelastic media containing random configurations of spherical scatterers, *J. Acoust. Soc. Am.*, 131 (2), 2012, pp. 1113–1120.
- [16] Xu Y., Fast evaluation of the Gaunt coefficients, *Mathematics of Computation of the American Mathematical Society* 65, no. 216, 1996, pp. 1601–1612.
- [17] Krishnamoorthy K., *Handbook of Statistical Distributions with Applications*, Chapman & Hall, 2006.
- [18] Torquato S., *Random heterogeneous materials*, Springer, 2002, pp. 160–161.
- [19] Kotlarczyk M., Chen S.-H., Analysis of small angle neutron scattering spectra from polydisperse interacting colloids, *J. Chem. Phys.* 79, 1983, 2461.
- [20] Pandey R. K., Tripathi D. N., Schulz distribution function and the polydispersity of the binary suspension of charged macroions, *Colloids and Surfaces A: Physicochemical and Engineering Aspects* 190, 2001, pp. 217–227.
- [21] Gazzillo D., Giacometti A., Guido Della Valle R., Venuti E., Carsughi F., A scaling approximation for structure factors in the integral equation theory of polydisperse nonionic colloidal fluids, arXiv: cond-mat/9907359v3 [cond-mat.stat-mech] 15 Oct 1999.
- [22] Linton C. M., Martin P. A., Multiple scattering by random configurations of circular cylinders: Second-order corrections for the effective wavenumber, *J. Acoust. Soc. Am.* 117 (6), 2005, pp. 3413–3423.
- [23] Aristégui C., Angel Y. C., Effective mass density and stiffness derived from P-wave multiple scattering, *Wave Motion*, vol. 44, 2007, pp. 153–164.
- [24] Kuster G. T., Toksöz M. N., Velocity and attenuation of seismic waves in two-phase media: Part I. Theoretical Formulation; Part II: Experimental Results, *Geophysics* 39, 1974, pp. 587–618.
- [25] Stoll R. D., Kan T. K., Reflection of acoustic waves at a water-sediment interface, *J. Acoust. Soc. Am.* 70, 1981, pp. 149–156.
- [26] Abramowicz M., Stegun I. A., *Handbook of Mathematical Functions*, Dover New York, 1965, pp. 363.
- [27] Luppé F., Conoir J.-M., Franklin H., Scattering by a fluid cylinder in a porous medium: application to trabecular bone, *J. Acoust. Soc. Am.* 111 (6), 2002, pp. 2573–2582.