Appendix B : Approximate invariant means for boundary actions of hyperbolic groups

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We present here a different and, we believe, easier way of proving the following theorem of Adams:

Let Γ be a discrete word hyperbolic group and denote $\partial\Gamma$ its Gromov boundary, then Γ acts on $\partial\Gamma$ in an amenable way (as in the definition of this article)

As a corrolary, we get that the reduced C*-algebra of Γ is exact and also the result of Kuhn and Steger of the weak containment of the boundary action in the regular representation.

We prove this theorem à la Day, showing that there exists a sequence of positive compactly supported borel functions f_n on $\Gamma \times \partial\Gamma$ such that

$$\forall x \in \partial\Gamma \qquad \int_\Gamma |f_n(g,x)|dg > 0$$

and

$$\lim_{n\to\infty} \left(\sup_{x\in\partial\Gamma} \frac{\int_\Gamma |f_n(g,x) - h.f_n(g,x)|dg}{\int_\Gamma |f_n(g,x)|dg} \right) = 0 \qquad \forall h \in \Gamma$$

where $(h.f)(g,x) = f(h^{-1}g, h^{-1}.x)$ is the action induced by the diagonal action of Γ in $\Gamma \times \partial\Gamma$.

The construction of the f_n's is simple: we just average characteristic functions of geodesic rays in the $x \in \partial\Gamma$ direction. The amenability condition is then a corollary of a geometric lemma that states that all rays are at some point in a δ-neighborhood of each other, regardless of the starting point, provided they all point in the same direction.

1 Some background on hyperbolic groups

Let Γ be a finitely generated discrete group. Associated to the set of generators, we have a left invariant word length ℓ, therefore a distance d may be defined by $d(a,b) = \ell(a^{-1}b)$ and the left translation action of the group on itself is isometric.

For the distance, the group is proper (all ball of radius R have the same finite number of elements) and geodesic.

To fix notations a geodesic g between two points a and b is an isometry $g : [0, d(a,b)] \to \Gamma$ with $g(0) = a$ and $g(d(a,b)) = b$. We will call $[[a,b]]$ the set of all geodesics between a and b (it may not be a singleton) and a geodesic ray an isometry from $\mathbf{N}^+$ to Γ.

There exist several different, but equivalent, ways of expressing the property of hyperbolicity; in a very geometric approach, we could say that Γ is hyperbolic for some constant δ if any geodesic triangle is δ-thin, meaning that any geodesic edge is in a δ-neighborhood of the union of the other two.

But to do computations, we need a more practical definition, we shall take the following:

Definition 1.1 *We say that* Γ *is* δ*-word hyperbolic if for any four group elements* x, y, z, t *we have*

$$d(x,y) + d(z,t) \leq \max(d(x,z) + d(y,t), d(x,t) + d(y,z)) + 2\delta$$

There are several way of describing the boundary (See [3] Chapter 2). Either we can consider the set of all sequences of points in Γ such that

$$\lim_{n,m\to\infty} d(x_n, a) + d(x_m, a) - d(x_n, x_m) = +\infty$$

where a is a base point together with the equivalence relation $(x_n) \equiv (y_n)$ iff $\lim_{n\to\infty} d(x_n, a) + d(y_n, a) - d(x_n, y_n) = +\infty$. It describes a compact space $\partial\Gamma$ whose definition is independent of the choice of a. Or we can consider the set of all geodesic rays starting from some point a, and endows it with the uniform topology on compact sets. We prescribe an equivalence relation by identifying two rays r_1 and r_2 if $n \mapsto d(r_1(n), r_2(n))$ is a bounded map on $\mathbf{N}$. What we get is a compact and metrizable space $\partial\Gamma_a$. Because the hyperbolic space Γ is proper, the map from rays r to sequences $(r(n))_{n\geq 0}$ is a homeomorphism and identifies all the spaces $\partial\Gamma_a$.

For any group element g, the left translation by g induces a homemorphism between $\partial\Gamma_e$ and $\partial\Gamma_g$, and through the above identification an action of Γ on $\partial\Gamma$ that we will denote by $x \mapsto g.x$ for x in $\partial\Gamma$.

Given $(a, x) \in \Gamma \times \partial\Gamma$, $[[a, x[[$ will denote the set of all geodesics from a to x, i.e. the isometries from $\mathbf{N}^+$ to Γ starting in a and such that it defines the boundary point x. By the above definition of the action, it is clear that given a geodesic ray r in $[[a, x[[$ and a group element g, the left translated geodesic ray $g.r$ is a geodesic in $[[ga, g.x[[$.

Hyperbolicity of the group has some consequences for the set $\Gamma \cup \partial\Gamma$. For example we get that any geodesic triangle (with the extension explained above) is 24δ-thin for the metric of the group (excluding the end points) or that any two geodesics between the same two points are in a 8δ-neighborhood of each other.

2 A geometric property

Lemma 2.1 *Let* K *be an integer, there exists an* $0 < M \leq K + 48\delta$ *such that for all points* a, b *in* Γ *and* $x \in \Gamma \cup \partial\Gamma$ *with* $d(a,b) < K$ *we have that for all points* p *in a geodesic from* a *to* x *and* q *in a geodesic from* b *to* x *with* $d(a,p) = d(b,q)$ *the relation* $d(p,q) < M$.

Since all triangles are 24δ-thin and due to the symmetry in p and q, we have that either p is at distance at most 24δ from a point q_0 on a geodesic from b to x containing q or that both p and q are at distance 24δ from a geodesic between

a and b. In the latter case we obviously have $d(p,q) < 48\delta + d(a,b)$, hence the result. Whereas in the former case, we have

$$|d(b,q_0) - d(a,p)| < K + 24\delta$$

using the triangle inequality. Hence $d(q,q_0) < K + 24\delta$, using $d(a,p) = d(b,q)$. Thus $d(p,q) < K + 48\delta$.

Lemma 2.2 *Let K, L be integers, with L greater than $3K + \delta$, a, b, e, f be four points in Γ such that $d(a,b) < K$, $d(e,f) < K$, assume furthermore that $d(a,e) > 3L$ and $d(b,f) > 3L$ then for all geodesic g_0 between a and e and all geodesic g between b and f, any point p of the segment $g([L,2L])$ is at a distance at most 4δ of a point q in $g_0([L-K, 2L+K])$ such that $d(b,p) = d(b,q)$.*

Having chosen our 6 points a, b, e, f, p, q according to the assumptions, we will repeatedly use the hyperbolic inequality.

Let's prove first that $d(p,q) < 2\delta + 4K$.

Considering the 4 points b, p, q, f, we have

$$d(b,f) + d(p,q) < 2\delta + \max(d(b,p) + d(q,f), d(b,q) + d(p,f))$$

hence $d(p,q) < 2\delta + d(q,f) - d(p,f))$

Obviously $d(q,f) \leq K + d(q,e)$ and $d(a,q) \leq d(b,q) + K$, now $d(p,f) = d(b,f) - d(b,p) = d(b,f) - d(b,q)$, so $d(p,f) \geq d(b,f) - d(a,q) - K$. Therefore $d(q,f) - d(p,f) < 2K + (d(a,e) - d(b,f))$. But we also have $|d(a,e) - d(b,f)| < 2K$, and as a consequence $d(p,q) < 2\delta + 4K$.

So far we know that $d(p,q) < 2\delta + \max(d(b,q) - d(b,p), d(q,f) - d(p,f))$, but using the symetry between the letters, we also have $d(p,q) < 2\delta + \max(d(a,p) - d(a,q), d(p,e) - d(q,e))$.

Let's consider now the points a, p, b, q:

$$d(a,p) + d(b,p) < 2\delta + \max(d(a,q) + d(b,p), d(a,b) + d(p,q)).$$

The first term of the max is greater than $2L - K$ and the second is at most $2\delta + 5K$. Since $L > \delta + 3K$, the first term is greater than the second, so

$$d(a,p) + d(b,q) < 2\delta + d(a,q) + d(b,p)$$

or $d(a,p) - d(a,q) < 2\delta$.

Writing the same inequality for the points p, e, q, f, and for the same reason we get $d(p,e) - d(q,e) < 2\delta + d(p,f) - d(q,f))$.

So $d(p,q) < 2\delta + \max(2\delta, 2\delta + d(p,f) - d(q,f))$ Hence either $d(p,q) < 2\delta$ by our first inequality or $d(p,q) < 4\delta$ depending on the sign of $d(p,f) - d(q,f)$.

Lemma 2.3 *Let K be an integer and assume $L > 3K + 150\delta$ then for any two points a, b in Γ with $d(a,b) < K$ and x in $\partial\Gamma$ and for all geodesic g_0 from a to x and g from b to x, any point p of $g([L, 2L])$ is at distance at most 4δ from a point q in $g_0([L-K, 2L+K])$.*

It follows easily from the previous two lemmas. By hypothesis L is greater than $\delta + 3\sup(K, M)$ since we can take $M = K + 48\delta$ by lemma 2.1. Then apply lemma 2.2 with $e = g_0(3L)$ and $f = g(3L)$.

3 An averaging construction

Let $a \in \Gamma$ and $x \in \partial\Gamma$, and consider for any positive integer k $I(a,x,k) = \{g \in [[a_1, x[[,\ d(a_1, a) < k\}$ the set of all geodesics pointing to the direction x and starting not to far from the point a. Choosing a length $l > 0$, we define

$$F(a,x,k,l) = \text{characteristic function of } \bigcup_{g \in I(a,x,k)} g([l, 2l])$$

to be large portions of these geodesics, far enough from our reference point. Finally we set

$$H(a,x,l) = \frac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} F(a,x,k,l)$$

to be our ad hoc average.

For F a compactly supported function on Γ, we will note $||F||$ its norm in $\ell^1(\Gamma)$.

Proposition 3.1 *We have the following*

1. $||H(a,x,l)|| \geq l \quad \forall a \in \Gamma, \forall x \in \partial\Gamma$
2. $(x,t) \to H(a,x,l)(t)$ *is upper continuous, a and l fixed*
3. $\sup_{x \in \partial\Gamma} ||H(ga,x,l) - H(a,x,l)|| = o(l)$ *for $g, a \in \Gamma$ fixed.*

The first property is obvious for $F(a,x,k,l)$ is always greater than the characteristic function of a geodesic of length l, therefore $||H(a,x,l)|| \geq l$.

And the third follows from the lemma:

Lemma 3.2 *For all positive integer c and $a \in \Gamma$, we have*

$$\sup_{x \in \partial\Gamma} \left(\sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)|| \right) = O(l)$$

First note that $k \mapsto F(a,x,k,l)$ is increasing, so

$$\sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)|| = \sum_{k<\sqrt{l}} ||F(a,x,k+c,l)|| - ||F(a,x,k,l)||$$

and is therefore less that $\sum_{\sqrt{l}\leq k<\sqrt{l}+c} ||F(a,x,k,l)||$.

For l large enough we have $3(\sqrt{l}+c)+150\delta < l$, hence lemma 2.3 applies and $F(a,x,k,l)$ is in a 4δ-neighborhood of a geodesic of length $l+2(\sqrt{l}+c)$.

Then $\sum_{\sqrt{l}\leq k<\sqrt{l}+c} ||F(a,x,k,l)|| \leq c(l+2(\sqrt{l}+c))B$ where B is the number of points in (any) ball of radius 4δ.

Going back to point 3, let $c = d(ga,a)$. We certainly have

$$\begin{aligned} ||H(ga,x,l) - H(a,x,l)||_{L^1(\Gamma)} &\leq \tfrac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)|| + \\ &\tfrac{1}{\sqrt{l}} \sum_{k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k,l)|| \end{aligned}$$

But we know by the preceding lemma that the second term is $O(l^{\frac{1}{2}})$ uniformly in x whereas the first term splits into

$$\frac{1}{\sqrt{l}} \sum_{0\leq k\leq c} ||F(a,x,k+c,l) - F(ga,x,k,l)||$$

which is bounded by

$$\frac{c}{\sqrt{l}}(||F(a,x,2c,l)|| + ||F(ga,x,c,l)||)$$

and

$$\frac{1}{\sqrt{l}} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(ga,x,k,l)||$$

which is not greater than

$$\frac{1}{\sqrt{l}} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k-c,l)||$$

since for $k > c$, one has $F(a,x,k-c,l) \leq F(ga,x,k,l) \leq F(a,x,k+c,l)$.

By lemma 2.1 for $K = 2c$, there exists M, independent from x, such that $F(a,x,2c,l)$ is less than the characteristic function of a tube of radius M along a geodesic of length l, therefore there exists a constant C such that, for any $(a,x) \in \Gamma \times \partial\Gamma$, $|F(a,x,2c,l)| \leq C.l$.

Hence $\frac{c}{\sqrt{l}} \sup_{x\in\partial\Gamma} (||F(a,x,2c,l)|| + ||F(ga,x,c,l)||)$ is $o(l)$ as well as

$$\frac{1}{\sqrt{l}} \sup_{x\in\partial\Gamma} \sum_{c<k<\sqrt{l}} ||F(a,x,k+c,l) - F(a,x,k-c,l)||$$

by lemma 3.2.

Finally for point 2 of proposition 3.1, it suffices to prove that for k, l, a fixed, $(x,t) \to F(a,x,k,l)(t)$ has a local maximum everywhere. This follows easily, since Γ is discrete, from

Lemma 3.3 *Let r_1, r_2 be integers and a in Γ, x in $\partial\Gamma$ then there exists a neighborhood V of x, such that for all $x' \in V$*

$$\left(\bigcup_{g\in[[a,x'[[} g([r_1,r_2])\right) \subset \left(\bigcup_{g\in[[a,x[[} g([r_1,r_2])\right)$$

Since we are considering geodesics, we only need to prove that for all $k \leq r$, there exists a neighborhood V of x, such that for all $x' \in V$

$$\left(\bigcup_{g\in[[a,x'[[} g(k)\right) \subset \left(\bigcup_{g\in[[a,x[[} g(k)\right)$$

By contradiction, assuming the opposite, we have a sequence g_n of geodesics (the topology of $\partial\Gamma$ is metrizable) with end points converging to x and such that $g_n(k)$ is not contained in $\bigcup_{g\in[[a,x[[} g(k)$, which is finite since it is contained in $B(a,r)$, thus open and compact since Γ is discrete.

By Arzela-Ascoli there exists a subsequence that converges uniformly on compact sets to a geodesic g_∞ from a to x, and $g_\infty(k)$ is not in $\bigcup_{g\in[[a,x[[} g(k)$. Hence a contradiction.

We have now all the elements for the theorem:

Theorem 3.4 *Let Γ be a discrete hyperbolic group and $\partial\Gamma$ its (Gromov) boundary. Then Γ acts in a amenable way on $\partial\Gamma$.*

Let $f_n(g,x) = H(e,x,n)(g)$, then f_n is positive, compactly supported (the support is contained in $B(e,3n) \times \partial\Gamma$), Borel and

$$\forall x \in \partial\Gamma, \quad ||H(e,x,n)||_{L^1(\Gamma)} \geq n$$

therefore

$$\forall x \in \partial\Gamma, \forall n > 0 \quad \int_\Gamma |f_n(g,x)|dg > 0.$$

Finally $h.f(g,x) = H(e,h^{-1}.x,n)(h^{-1}g) = H(h,x,n)(g)$. Then Proposition 3.1 gives

$$\lim_{n\to\infty}\left(\sup_{x\in\partial\Gamma} \frac{\int_\Gamma |f_n(g,x) - h.f_n(g,x)|dg}{\int_\Gamma |f_n(g,x)|dg}\right) =$$

$$= \lim_{n\to\infty} \left(\sup_{x\in\partial\Gamma} \frac{||H(e,x,n) - H(h,x,n)||_{L^1(\Gamma)}}{||H(e,x,n)||_{L^1(\Gamma)}} \right) \leq$$
$$\leq \lim_{n\to\infty} \left(\frac{o(n)}{n} \right) = 0$$

Corollary 3.5 *Let Γ be an hyperbolic group. Then $C_r^*(\Gamma)$ is exact.*

Indeed since $\partial\Gamma$ is compact, we have an imbedding of $C_r^*(\Gamma)$ into the reduced cross product of $C(\partial\Gamma)$ by Γ and this algebra is nuclear since Γ acts in an amenable way.

Corollary 3.6 *(See [8] TH 1.X) Let Γ be an hyperbolic group, then the boundary action is weakly contained in the regular representation.*

The boundary action yields a representation of the (full) cross product of $C(\partial\Gamma)$ by Γ, but this algebra is isomorphic to the reduced cross product by theorem 3.4. Therefore we get by composition a representation of $C_r^*(\Gamma)$.

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