Fractional powers on noncommutative L p for p

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FRACTIONAL POWERS ON NONCOMMUTATIVE L_p FOR $p < 1$

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Abstract. We prove that the homogeneous functional calculus associated to $x \mapsto |x|^\theta$ or $x \mapsto \mathrm{sgn}\,(x)|x|^\theta$ for $0 < \theta < 1$ is θ-Hölder on selfadjoint elements of noncommutative L_p-spaces for $0 < p \leqslant \infty$ with values in $L_{p/\theta}$. This extends an inequality of Birman, Koplienko and Solomjak also obtained by Ando.

1. Introduction

This note deals with the perturbation theory of functional calculus of selfadjoint operators on Hilbert spaces. More precisely, given a function $f : \mathbb{R} \to \mathbb{R}$, the problem is to get a control of $\|f(x) - f(y)\|_{\mathfrak{S}}$ for some symmetric norm on selfadjoint operators in terms of possibly another norm $\|x - y\|_{\mathfrak{S}'}$. This topic was developed from the 50's by the Russian school. Birman and Solomjak had a strong impact on it by the introduction of operator integrals in the 60's. Since then, this subject has been very active. Many mathematicians tried to enlarge the classes of functions f or norms involved. The list would be too long, but we can quote Arazy [5, 6], Ando [4], and more recently the breakthroughs by Alexandrov-Peller [1, 2],... and Potapov-Sukochev [21, 20] and their coauthors [22],... Usually the results are stated for symmetric (quasi-)norms on compact operators, for instance the Schatten p-classes S^p for $0 < p \leqslant \infty$. Nevertheless the noncommutative integration theory in von Neumann algebras also gives a natural framework to study these questions.

Our starting point is an inequality in [8], for any fully symmetric norm $\|.\|_{\mathfrak{S}}$ and any $0 < \theta < 1$, and x, y positive operators on some Hilbert space, i.e $x, y \in B(H)^+$

$$\left\|x^\theta - y^\theta\right\|_{\mathfrak{S}} \leqslant \left\|\,|x - y|^\theta\,\right\|_{\mathfrak{S}}.$$

It was extended by Ando [4] to any operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ instead of $x \mapsto x^\theta$. Dodds and Dodds [13] adapted the proof to semi-finite von Neumann algebras for all fully symmetric norms.

In the case of Schatten classes, Birman Koplienko and Solomjak's result or Ando's proof actually give that for $p \geqslant \theta$ and $x, y \in B(H)^+$

$$\left\|x^\theta - y^\theta\right\|_{p/\theta} \leqslant \left\|\, x - y\,\right\|_p^\theta.$$

This also holds for semi-finite Neumann algebras by [13] or [20]. For general von Neumann algebras, Kosaki got the case $p = \theta$ in [15] with an extra factor, a full argument can be found in [11] or [24].

Another remarkable extension was obtained in [2] for Schatten p-classes when $1 < p < \infty$; it is shown that for any θ-Hölder function f on $\mathbb{R}$, with $0 < \theta < 1$ and any selfadjoint $x, y \in B(H)^{sa}$, one has

$$\|f(x) - f(x)\|_{p/\theta} \leqslant C_{p,f}\|x - y\|_p^\theta. \qquad (1)$$

In particular this holds if $f(x) = |x|^\theta$ or $f(x) = \mathrm{sgn}(x)|x|^\theta$. For them, the arguments can be adapted to general von Neumann algebras [24] and one can also reach $p = 1$ in (1).

Surprisingly, when $p < 1$ even for Schatten classes, very little is known. One can find some asymptotic estimates in [8] or [26] but (1) seems to be unknown. Weaker related inequalities were also recently obtained in [27].

Among general results Raynaud [23] proved that $x \mapsto f(x)$ from L_p to $L_{p/\theta}$ is uniformly continuous on balls for f as above. In [18], for type II von Neumann algebras a strange quantitative estimate was obtained for its modulus of continuity.

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Our main result is that (1) holds for all $0 < p \leqslant \infty$ for both $f(x) = |x|^\theta$ and $f(x) = \mathrm{sgn}(x)|x|^\theta$ and all von Neumann algebras. We hope that the techniques developed here may be useful for related topics.

We do not address similar questions when $\theta > 1$. When $p \geqslant 1$, this is done in [24] and when $p < 1$, some local results can be found in [5, 22] for Schatten classes.

As usual to deal with such questions, one has to find norm estimates for some Schur multipliers, this is done in the second section. Next, they are used to derive the main result for semi-finite von Neumann algebras. The argument heavily rests on homogeneity of f. To generalize to type III algebras, one usually relies on the Haagerup reduction principle, but it involves approximations using conditional expectations that are not bounded when $p < 1$ and it seems difficult to use it in our situation. The only available tool we have is to use weak-type inequalities in semi-finite algebras to go to type III. The situation is so particular here that this can be done quite easily in section 4. We end up with some general remarks and extensions.

In the whole paper, we freely use noncommutative L_p-spaces. One may use [19, 17, 29] and [14] as general references. When τ is a normal faithful trace on a von Neumann algebra $\mathcal{M}$, we use the classical definition of noncommutative L_p associated to $\mathcal{M}$

$$L_p(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}, \tau) \mid \|x\|_p^p = \tau(|x|^p) < \infty\},$$

where $L_0(\mathcal{M}, \tau)$ is the space τ-measurable operators (see [29]). When dealing with more general von Neumann algebras, we rely on Haagerup's construction. Given a normal faithful semi-finite weight φ_0 on a von Neumann algebra $\mathcal{M}$, Haagerup defined the noncommutative L_p-space $L_p(\mathcal{M}, \varphi_0)$ for $0 < p \leqslant \infty$ (see [29]). His definition is independent of φ_0 (Corollary 38 in [29]). When φ_0 is a normal faithful trace, his definition is equivalent to the previous one (up to a complete isometry) but the identifications are not obvious. Nevertheless for most of our statements, we won't need the reference to φ_0 or τ so we may simply write $L_p(\mathcal{M})$. When $0 < p < 1$, $L_p(\mathcal{M})$ is a p-normed space so that for all families (a_k) in $L_p(\mathcal{M})$, $\| \sum_{k=1}^n a_k\|_p^p \leqslant \sum_{k=1}^n \|a_k\|_p^p$.

We will use the notation $S^p_{I,J}$ for the Schatten p-class on $B(\ell_2(J), \ell_2(I))$, this is naturally a subspace of $L_p(B(\ell_2(I \cup J)), \mathrm{tr})$, where tr is the usual trace. Thus by $S^p_{I,J}[L_p(\mathcal{M})]$, we will mean the corresponding subspace of $L_p(B(\ell_2(I \cup J)) \otimes \mathcal{M}, \mathrm{tr} \otimes \varphi_0)$, one can think of it as matrices indexed by $I \times J$ with coefficients in $L_p(\mathcal{M})$. We will often use non countable sets like $I =]0, 1]$.

As usual, we denote constants in inequalities by C_{p_i} if they depend only on parameters (p_i). They may differ from line to line.

2. Schur multipliers

A Schur multiplier with symbol $M = (m_{i,j})_{i \in I, j \in J}$ over $M_{I,J}$, the set of matrices indexed by sets I and J, is formally given by

$$S_M((a_{i,j})_{i \in I, j \in J}) = M \circ A = (m_{i,j} a_{i,j})_{i \in I, j \in J}.$$

Definition 2.1. *Given a matrix $M = (m_{i,j})_{i \in I, j \in J}$ of complex numbers, we say that M defines a p-completely bounded Schur multiplier for some $0 < p \leqslant \infty$ if the map $S_M \otimes Id_{L_p(\mathcal{M})}$ on $S^p_{I,J}[L_p(\mathcal{M})]$ is bounded for all von Neumann algebra $\mathcal{M}$ and we put $\|M\|_{pcb} = \sup_{\mathcal{M}} \|S_M \otimes Id_{L_p(\mathcal{M})}\|$.*

Remark 2.2. For $1 < p \neq 2 < \infty$, this is not exactly the usual definition of complete boundedness but it is formally stronger. Indeed an unpublished result of Junge states that $\|S_M \otimes Id_{L_p(\mathcal{M})}\| \leqslant \|S_M\|_{cb}$ if $\mathcal{M}$ is a QWEP von Neumann algebra.

We start by easy examples that can be found in [1].

Lemma 2.3. *Let $(\alpha_k) \in \ell_p(\mathbb{Z})$ for $0 < p \leqslant 1$ and assume that $(f_k) \in \ell_\infty(I)^{\mathbb{Z}}$ and $(g_k) \in \ell_\infty(J)^{\mathbb{Z}}$ are bounded families. Then M given by $m_{i,j} = \sum_k \alpha_k f_k(i) g_k(j)$ is a p-completely bounded Schur multiplier with*

$$\|M\|_{pcb} \leqslant \|(\alpha_k)\|_p \sup_k \|f_k\|_\infty . \|g_k\|_\infty.$$

Proof. It is clear that a rank one symbol $M_k = (f_k(i) g_k(j))_{i \in I, j \in J}$ defines a p-completely bounded Schur multiplier with norm $\|f_k\|_\infty . \|g_k\|_\infty$ for all p and k. The result then follows by the p-triangular inequality. □

We will often use permanence properties of *pcb*-Schur multipliers.

Lemma 2.4. *Assume $M=(m_{i,j})_{i\in I,j\in J}$ is a p-completely bounded Schur multiplier, then*

- $M'=(m_{i,j})_{i\in I',j\in J'}$ *with $I'\subset I$, $J'\subset J$, then $\|M'\|_{pcb}\leqslant\|M\|_{pcb}$.*
- $M'=(m_{i,j})_{(i,k)\in I\times K,(j,l)\in J\times L}$ *for any non empty sets K,L, then $\|M'\|_{pcb}=\|M\|_{pcb}$.*

Proof. We view $\ell_2(I')$, $\ell_2(J')$ as subspaces of $\ell_2(I)$, $\ell_2(J)$. Let $P=(1_{I'}(i)1_{J'}(j))_{i\in I,j\in J}$, this is a rank one p-completely bounded Schur multiplier with norm 1, and $S_{M'}$ coincides with the restriction of $S_P\circ S_M$ to matrices indexed by $I'\times J'$.

The second point is classical using tensorisation with $B(\ell_2(L),\ell_2(K))\otimes\mathcal{M}$ instead of $\mathcal{M}$. □

Since finitely supported matrices are dense in $S^p_{I,J}[L_p(\mathcal{M})]$, we also have

Lemma 2.5. *If $(M_n)\in M_{I,J}^{\mathbb{N}}$ is a bounded sequence of pcb-Schur multipliers converging pointwise to some M, then M is also a pcb-Schur multiplier with $\|M\|_{pcb}\leqslant\lim\|M_n\|_{pcb}$.*

Remark 2.6. A p-completely bounded Schur multiplier M for $p\leqslant 1$ is automatically q-completely bounded for $p<q\leqslant\infty$. Indeed, the extreme points of the unit ball of S^1 are rank one matrices, but for those matrices the S^1 and S^p norms coincide. Thus S_M must be bounded on S^1. But bounded Schur multipliers on S^1 are automatically 1-completely bounded (see [17]). Thus we get the result on S^q for all $1\leqslant q\leqslant\infty$ by complex interpolation and duality. The case $p<q\leqslant 1$ also follows by interpolation.

The following is a suitable adaptation of classical arguments (see [8, 25, 26]). We use the measured space $L_2([0,2\pi]^2,\frac{1}{(2\pi)^2}dm_2)$ where m_2 is the Lebesgue measure.

Lemma 2.7. *Let $K:[0,2\pi]\times[0,2\pi]\to\mathbb{C}$ be a 2π-periodic continuous function such that for any $d\geqslant 0$ $\frac{\partial^{d+1}}{(\partial y)^d\partial x}K$ is continuous. Then $M=(K(x,y))_{x,y\in[0,2\pi]}$ is a p-completely bounded Schur multiplier for all $0<p\leqslant 1$ with for $d>1/p$*

$$\|M\|_{pcb}\leqslant C\Big(\frac{2}{dp-1}+2\Big)^{1/p}\Big(\big\|\frac{\partial^{d+1}}{(\partial y)^d\partial x}K\big\|_2+\big\|\frac{\partial^d}{(\partial y)^d}K\big\|_2+\big\|\frac{\partial}{\partial x}K\big\|_2+\big\|K\big\|_2\Big),$$

where C is a universal constant. Moreover if $M_i=(K_i(x,y))_{x,y\in[0,2\pi]}$ is a family of matrices as above, indexed by $i\in I$, then

$$M=(K_i(x,y))_{(x,i)\in[0,2\pi]\times I,y\in[0,2\pi]}$$

or its transpose satisfies

$$\|M\|_{pcb}\leqslant C\Big(\frac{2}{dp-1}+2\Big)^{1/p}\sup_i\Big(\big\|\frac{\partial^{d+1}}{(\partial y)^d\partial x}K_i\big\|_2+\big\|\frac{\partial^d}{(\partial y)^d}K_i\big\|_2+\big\|\frac{\partial}{\partial x}K_i\big\|_2+\big\|K_i\big\|_2\Big).$$

Of course, this is relevant only if the above sup is finite.

Proof. We rely on Fourier expansions, put $e_k(x)=e^{\mathrm{i}kx}$ and $h_{k,l}(x,y)=e_k(x)e_l(y)$. As $(h_{k,l})_{k,l\in\mathbb{Z}}$ is an orthonormal basis in $L_2([0,2\pi]^2,\frac{1}{(2\pi)^2}dm_2)$, we have the equality in L_2, $K=\sum_{l,k\in\mathbb{Z}}\alpha_{k,l}h_{k,l}$ where $\alpha_{k,l}=\langle K,h_{k,l}\rangle=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}K(x,y)e^{-\mathrm{i}kx}e^{-\mathrm{i}ly}dydx$.

Assume for the moment that $l\neq 0$. Integrating by part in y, we get for $d\geqslant 0$, $\alpha_{k,l}=\frac{1}{(\mathrm{i}l)^d}\langle\frac{\partial^d}{(\partial y)^d}K,h_{k,l}\rangle$. Let $\beta_{k,l}=(\mathrm{i}l)^d\alpha_{kl}$. When $k\neq 0$, another integration by part with respect to x gives $\beta_{k,l}=\frac{1}{\mathrm{i}k}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\partial^{d+1}}{(\partial y)^d\partial x}K(x,y)e^{-\mathrm{i}kx}e^{-\mathrm{i}ly}dydx$. The Cauchy-Schwarz inequality gives that

$$\sum_{k\neq 0}|\beta_{k,l}|\leqslant\Big(\sum_{k\neq 0}\frac{1}{k^2}\Big)^{1/2}\Big(\sum_{k\neq 0}\big|\langle\frac{\partial^{d+1}}{(\partial y)^d\partial x}K,h_{k,l}\rangle\big|^2\Big)^{1/2}\leqslant C\Big\|\frac{\partial^{d+1}}{(\partial y)^d\partial x}K\Big\|_2.$$

Thus $\sum_{k\in\mathbb{Z}}|\beta_{k,l}|\leqslant C\big\|\frac{\partial^{d+1}}{(\partial y)^d\partial x}K\big\|_2+\big\|\frac{\partial^d}{(\partial y)^d}K\big\|_2=C_d$ independent from l and one can define a continuous function $f_l(x)=\sum_{k\in\mathbb{Z}}\frac{1}{\mathrm{i}^d}\beta_{k,l}e_k(x)$ bounded by C_d.

In the same way to deal with $l=0$, $f_0(x)=\sum_{k\in\mathbb{Z}}\alpha_{k,0}e_k(x)$ is a continuous function. Indeed as above

$$\sum_{k\in\mathbb{Z}}|\alpha_{k,0}|\leqslant\Big(\sum_{k\neq 0}\frac{1}{k^2}\Big)^{1/2}\Big(\sum_{k\neq 0}\big|\langle\frac{\partial}{\partial x}K,h_{k,0}\rangle\big|^2\Big)^{1/2}+|\alpha_{0,0}|,$$

f_0 is bounded by $C_0=C\big\|\frac{\partial}{\partial x}K\big\|_2+\big\|K\big\|_2$.

Choosing $d > \frac{1}{p}$, we can conclude to the pointwise equality

$$K(x,y) = f_0(x)e_0(y) + \sum_{l\neq 0} \frac{1}{l^d} f_l(x)e_l(y). \tag{2}$$

The result follows directly from Lemma 2.3 by choosing $I = \mathbb{Z}$, $\alpha = (1_{k\neq 0}\frac{1}{k^d} + 1_{k=0})_k \in \ell_p(\mathbb{Z})$, f_k as above and $g_k = e_k$. Obviously $\sup_k \|g_k\|_\infty = 1$, $\sup_k \|f_k\|_\infty \leqslant C_d + C_0$ and $\|\alpha\|_p \leqslant \left(\frac{2}{dp-1} + 2\right)^{1/p}$.

The second statement also follows from Lemma 2.3 since in (2), the factorization in y and the sequence α is independent from K_i. We have $K_i(x,y) = f_0^i(x)e_0(y) + \sum_{l\neq 0} \frac{1}{l^d} f_l^i(x)e_l(y)$, hence we can again take $\alpha = (1_{k\neq 0}\frac{1}{k^d} + 1_{k=0})_k \in \ell_p(\mathbb{Z})$, $f_k(x,i) = f_k^i(x)$ and $g_k(y) = e_k(y)$. We also have $\sup_k \|f_k\|_\infty \leqslant C_d + C_0$.

The same holds for the transpose of M as the condition in Lemma 2.3 is invariant by transposition. □

The Sobolev constant (of order d) for K will mean the quantity

$$\Big\|\frac{\partial^{d+1}}{(\partial y)^d \partial x} K\Big\|_2 + \Big\|\frac{\partial^d}{(\partial y)^d} K\Big\|_2 + \Big\|\frac{\partial}{\partial x} K\Big\|_2 + \|K\|_2.$$

Let $\theta \in]0,1[$, for $x, y \geqslant 0$ recall that

$$\frac{x^\theta - y^\theta}{x-y} = \int_0^1 \frac{\theta}{(tx + (1-t)y)^{1-\theta}} dt, \tag{3}$$

where the left hand side has to be understood as $\theta x^{\theta-1}$ if $x = y$.

Corollary 2.8. *The matrix* $N = \left(\frac{x^\theta - y^\theta}{x-y}\right)_{x\geqslant 0, y\in[1,2]}$ *defines a p-completely bounded Schur multiplier for* $0 < p \leqslant 1$ *with* $\|N\|_{pcb} \leqslant C_p$ *for some constant depending only on* p.

Proof. First we start by showing that $\left(\frac{x^\theta - y^\theta}{x-y}\right)_{0\leqslant x\leqslant 1/2, y\in[1,2]}$ is a *pcb*-Schur multiplier.

We fix a $\mathcal{C}^\infty$ function $\varphi : [-\pi,\pi] \to [0,1]$ with support in $[-1/4, 3/4]$ that is identically 1 on $[0,1/2]$ and another $\mathcal{C}^\infty$ function $\psi : [0,2\pi] \to [0,1]$ with support in $[7/8, 3]$ that is identically 1 on $[1,2]$. We define $K(x,y) = \varphi(x)\psi(y)\frac{1}{x-y}$ on $[-\pi,\pi] \times [0,2\pi]$. It is $\mathcal{C}^\infty$ and can be extended to a 2π-periodic $\mathcal{C}^\infty$ function. Thus Lemma 2.7 and a restriction yield that $\left(\frac{1}{x-y}\right)_{0\leqslant x\leqslant 1/2, y\in[1,2]}$ is a *pcb*-Schur multiplier. Then one just need to compose it with $\left(x^\theta - y^\theta\right)_{0\leqslant x\leqslant 1/2, y\in[1,2]}$ which is also clearly a *pcb*-Schur multiplier by Lemma 2.3.

Next we show that $\left(\frac{x^\theta - y^\theta}{x-y}\right)_{x\geqslant 1/2, y\in[1,2]}$ is also a *pcb*-Schur multiplier.

This time we fix a $\mathcal{C}^\infty$ function $\varphi : [0,2\pi] \to [0,1]$ with support in $[1/4, 3]$ that is identically 1 on $[1/2, 2]$.

For $i \geqslant 0$, one uses $K_i(x,y) = \varphi(x)\varphi(y)\frac{(x+i)^\theta - y^\theta}{(x+i)-y}$ on $[0,2\pi]^2$. It is clear that K_i can be extended to a $\mathcal{C}^\infty$ 2π-periodic function. By construction, for x and y in the support of φ and $t \in [0,1]$, the smallest value of $t(x+i) + (1-t)y$ is bigger than $1/4$. Thus, the formula (3) shows that any derivative of order l of $\frac{x^\theta - y^\theta}{x-y}$ on the support of K_i is bounded by $4^{l+1-\theta}\theta(1-\theta)...(l-\theta)$. Thus using the chain rule, one sees that K_i and its derivatives up to order $d+1$ are uniformly bounded independently of i and θ. Thus the same holds for the Sobolev constant in Lemma 2.7 for K_i.

Lemma 2.7 gives that $\big(K_i(x,y)\big)_{(x,i)\in[0,2\pi]\times\mathbb{N}, y\in[0,2\pi]}$ is *pcb*. By Lemma 2.4, we can conclude by restricting x to $[1/2, 3/2[\times\mathbb{N} \simeq [1/2,\infty[$ (via $(x,i) \mapsto x+i$) and y to $[1,2]$.

The Corollary follows by gluing the two pieces together. □

Corollary 2.9. *For* $k \in \mathbb{Z}$*, the matrix* $M_k = \left(\frac{x^\theta - y^\theta}{x-y}\right)_{x\geqslant 0, y\in[2^{-k-1}, 2^{-k}]}$ *is a p-completely bounded Schur multiplier for* $0 < p \leqslant 1$ *with* $\|M_k\|_{pcb} \leqslant C_p 2^{-k(\theta-1)}$ *for some constant depending only on* p.

Proof. This is obvious by homogeneity from Corollary 2.8 with a change of variables $x \leftrightarrow 2^{-k-1}x$, $y \leftrightarrow 2^{-k-1}y$. □

Remark 2.10. One can exchange the roles of x and y.

It will be convenient to redefine M_{-1}, gathering all $k\leqslant -1$:

Corollary 2.11. *The matrix* $M_{-1}=\Big(\frac{x^\theta-y^\theta}{x-y}\Big)_{x\geqslant 0, y\geqslant 1}$ *is a* p*-completely bounded Schur multiplier for* $0<p\leqslant 1$ *with* $\|M_{-1}\|_{pcb}\leqslant C_p\big(\frac{1}{1-\theta}\big)^{1/p}$ *for some constant depending only on* p.

Proof. Writing $[1,\infty[=\cup_{k\geqslant 0}[2^k,2^{k+1}[$ and using the previous Corollary for each piece, this follows from the p-triangular inequality as $(2^{k(\theta-1)})_{k\geqslant 0}\in \ell_p(\mathbb{N})$. Since $\|(2^{k(\theta-1)})_{k\geqslant 0}\|_p\leqslant c_p(1-\theta)^{-1/p}$ for some constant c_p. We get that $\|M_{-1}\|_{pcb}\leqslant c_pC_p(1-\theta)^{-1/p}$ where C_p comes from 2.9. □

Remark 2.12. The kernel in formula (3) is positive definite because for $x,y>0$ $\frac{x^\theta-y^\theta}{x-y}=c_\theta\int_{\mathbb{R}_+}t^\theta\frac{1}{x+t}\frac{1}{y+t}dt$ for some $c_\theta>0$. Using this fact and similar arguments, one can check that there is some C so that $\|M_{-1}\|_{pcb}\leqslant C$ for all $0<\theta<1$ and all $p\geqslant 1$.

We now turn to another family of multipliers.

Corollary 2.13. *For* $a\geqslant 1$*, the matrix* $H_a=\Big(\frac{1}{a+x+y}\Big)_{x,y\in[0,1]}$ *is a* p*-completely bounded Schur multiplier for* $0<p\leqslant 1$ *with* $\|H_a\|_{pcb}\leqslant C_p/a$ *for some constant* C_p *depending only on* p.

Proof. As for Corollary 2.8 take a smooth function $\varphi:[-\pi,\pi]\to[0,1]$ that is supported on $[-1/4,5/4]$ such that $\varphi(t)=1$ for $t\in[0,1]$. Define $K(x,y)=\frac{1}{a+x+y}\varphi(x)\varphi(y)$ on $[-\pi,\pi]$ and make it 2π-periodic so that it is $\mathcal{C}^\infty$. Using the chain rule, one easily sees that the Sobolev norms from Lemma 2.7 are dominated by C_p/a. □

Corollary 2.14. *Given* $a,b\geqslant 0$ *with* $a+b>0$*, one has*

$$\Big\|\Big(\frac{x^\theta\pm y^\theta}{x+y}\Big)_{x\geqslant a,y\geqslant b}\Big\|_{pcb}\leqslant C_p\big(\frac{1}{1-\theta}\big)^{1/p}\max\{a,b\}^{\theta-1},$$

for some constant C_p *depending on* p.

Proof. Without loss of generality we may assume $a\geqslant b$.

By a change of variable $x\leftrightarrow a(1+x)$ and $y\leftrightarrow a(t+y)$, with $t=b/a$, it boils down to show that $\Big\|\Big(\frac{(1+x)^\theta\pm(t+y)^\theta}{1+t+x+y}\Big)_{x\geqslant 0,y\geqslant 0}\Big\|_{pcb}$ is bounded independently of $t\in]0,1]$.

We use a dyadic decomposition related to $\max\{x,y\}$. Assume $x\in I_k=[2^k,2^{k+1}[$ and $y\in J_k=[0,2^k[$. Then by homogeneity and a change of variables $x\leftrightarrow 2^k(x+1)$, $y\leftrightarrow 2^ky$

$$\Big\|\Big(\frac{1}{1+t+x+y}\Big)_{x\in I_k,y\in J_k}\Big\|_{pcb}=2^{-k}\Big\|\Big(\frac{1}{1+2^{-k}(1+t)+x+y}\Big)_{x\in[0,1[,y\in[0,1[}\Big\|_{pcb}.$$

Setting $a=1+2^{-k}(1+t)\in[1,3]$ and using Corollary 2.13 the latter multiplier is bounded by C_p2^{-k}. The multiplier $\Big(((1+x)^\theta\pm(t+y)^\theta)1_{x\in I_k,y\in J_k}\Big)$ is bounded by a fixed multiple of $2^{k\theta}$. Thus $\Big\|\Big(\frac{(1+x)^\theta\pm(t+y)^\theta}{1+t+x+y}\Big)_{x\in I_k,y\in J_k}\Big\|_{pcb}\leqslant C_p2^{-k(1-\theta)}$. A similar estimate holds for the same symbol if $x\in J_{k+1}$ and $y\in I_k$ or $x,y\in[0,1[$ (with $k=0$). The sets $[0,1[^2$, $J_{k+1}\times I_k$, $I_k\times J_k$ for $k\geqslant 0$ form a partition of $[0,\infty[^2$ into product sets. Thus, the p-triangular inequality gives

$$\Big\|\Big(\frac{(1+x)^\theta\pm(t+y)^\theta}{1+t+x+y}\Big)_{x\geqslant 0,y\geqslant 0}\Big\|_{pcb}^p\leqslant C_p^p+2\sum_{k\geqslant 0}C_p^p2^{-p(1-\theta)k}\leqslant C_p^p\frac{1}{1-\theta}.$$

□

3. Ando's inequality in semi-finite algebras

Schur multipliers are intimately related to perturbations of the functional calculus of selfadjoint operators. Illustrations can be found in [9, 5, 2, 1, 21, 20, 22] and many other references.

Indeed let $f:\mathbb{R}\to\mathbb{R}$ be a (continuous) function. Assume that x,y are selfadjoint elements in some semi-finite von Neumann algebra $(\mathcal{M},\tau)$ with finite discrete spectra $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ and associated spectral projections $p_i\in\mathcal{M}$ and $q_j\in\mathcal{M}$. As $x=\sum_i x_ip_i$ and $y=\sum_j y_iq_j$, with $\sum_i p_i=\sum_j q_j=1$, one has

$$f(x)-f(y)=\sum_{i,j}p_i(f(x_i)-f(y_j))q_j=\sum_{i,j}p_i\frac{f(x_i)-f(y_j)}{x_i-y_j}(x-y)q_j, \tag{4}$$

where $\frac{f(u)-f(v)}{u-v}$ can take any value if $u = v \in \mathbb{R}$.

The map $T : z \mapsto \sum_{i,j} p_i \frac{f(x_i)-f(y_j)}{x_i-y_j} z q_j$ is very close to be a Schur multiplier with symbol the divided differences of f.

Lemma 3.1. *For any $0 < p \leqslant \infty$, let $M = (m_{i,j})$ be a p-completely bounded Schur multiplier then the following map $T_M : \mathcal{M} \to \mathcal{M}$*

$$T_M(z) = \sum_{i,j} m_{i,j} p_i z q_j$$

extends to a bounded map on $L_p(\mathcal{M})$ with $\|T_M\|_{L_p(\mathcal{M}) \to L_p(\mathcal{M})} \leqslant \|M\|_{pcb}$.

Proof. Recall that we assume that I and J are finite, $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ are orthogonal projections in $\mathcal{M}$ summing up to 1. The maps $\pi : \mathcal{M} \to B(\ell_2(J), \ell_2(I)) \otimes \mathcal{M}$ and $\rho : B(\ell_2(J), \ell_2(I)) \otimes \mathcal{M} \to \mathcal{M}$ defined by

$$\pi(z) = (p_i z q_j)_{i\in I, j\in J} = \begin{pmatrix} p_1 \\ \vdots \\ p_{|I|} \end{pmatrix} z \begin{pmatrix} q_1 & \dots & q_{|J|} \end{pmatrix}$$

$$\rho((X_{i,j})) = \sum_{i,j} p_i X_{i,j} q_j = \begin{pmatrix} p_1 & \dots & p_{|I|} \end{pmatrix} (X_{i,j}) \begin{pmatrix} q_1 \\ \vdots \\ q_{|J|} \end{pmatrix}$$

clearly extend to contractions at L_p-levels respectively on $\mathcal{M}$ and $B(\ell_2(J), \ell_2(I)) \otimes \mathcal{M}$ since the column and row matrices in the above products are contractions. Since $T_M = \rho \circ (S_M \otimes Id_{L_p(\mathcal{M})}) \circ \pi$, we get the lemma. □

We want to deal with homogeneous functions of selfadjoint operators, namely $f(x) = |x|^\theta$ or $f(x) = \mathrm{sgn}(x)|x|^\theta = x_+^\theta - x_-^\theta$.

We aim to prove the following theorem that we call Ando's inequality.

Theorem 3.2. *Let $0 < \theta < 1$ and $0 < p \leqslant \infty$ then there exists $C_{p,\theta}$ so that for any von Neumann algebra $\mathcal{M}$, and $x, y \in L_p(\mathcal{M})^{sa}$ one has*

$$\| |x|^\theta - |y|^\theta \|_{p/\theta} \leqslant C_{p,\theta} \|x - y\|_p^\theta, \qquad \|\mathrm{sgn}(x)|x|^\theta - \mathrm{sgn}(y)|y|^\theta\|_{p/\theta} \leqslant C_{p,\theta} \|x - y\|_p^\theta. \tag{5}$$

The reduction from type III to type II will be explained in the next section so we only deal with semi-finite algebras here.

Proof. The result for $p = \infty$ follows from Theorem 4.1 in [3].

The result for $f(x) = \mathrm{sgn}(x)|x|^\theta$ when $1 \leqslant p < \infty$ can be found in [24]. The absolute value map $x \mapsto |x|$ is bounded on $L_q(\mathcal{M})$ provided that $1 < q < \infty$. This is a result by Davies [12] for Schatten classes, that can be extended to all semi-finite von Neumann algebras (see Remark 6.2 in [10] for instance). Thus for $1 \leqslant p < \infty$, with $q = p/\theta$, we have an estimate $\| |x|^\theta - |y|^\theta \|_{p/\theta} \leqslant C_{p/\theta} \|\mathrm{sgn}(x)|x|^\theta - \mathrm{sgn}(y)|y|^\theta\|_{p/\theta}$. Therefore the Theorem holds for $f(x) = |x|^\theta$ using it for $f(x) = \mathrm{sgn}(x)|x|^\theta$ when $1 \leqslant p < \infty$.

We just need to prove the Theorem for $p < 1$.

If the result holds for the couples (p, θ_1) and $(p/\theta_1, \theta_2)$, it also holds for $(p, \theta_1\theta_2)$. Indeed for instance if

$$\| |x|^{\theta_1} - |y|^{\theta_1} \|_{p/\theta_1} \leqslant C_{p,\theta_1} \|x - y\|_p^{\theta_1} \quad \text{and} \quad \| |z|^{\theta_2} - |t|^{\theta_2} \|_{p/(\theta_1\theta_2)} \leqslant C_{p/\theta_1,\theta_2} \|z - t\|_{p/\theta_1}^{\theta_2},$$

then one gets with $z = |x|^{\theta_1}$, $t = |y|^{\theta_1}$:

$$\| |x|^{\theta_1\theta_2} - |y|^{\theta_1\theta_2} \|_{p/(\theta_1\theta_2)} \leqslant C_{p/\theta_1,\theta_2} C_{p,\theta_1}^{\theta_2} \|x - y\|_p^{\theta_1\theta_2}.$$

Hence $C_{p,\theta_1\theta_2} \leqslant C_{p/\theta_1,\theta_2} C_{p,\theta_1}^{\theta_2}$. By this transitivity, we reduce the proof to $p \leqslant \theta$; indeed if $p > \theta$, then $(p, \theta) = (p.1, p.\theta/p)$ and the estimate follows from that for (p, p) and $(1, \theta/p)$.

We do it only for $f(t) = |t|^\theta$ as the other case is similar.

We prove the inequality from case to case regarding $\mathcal{M}$ and the values of x and y.

Case 1: We assume that $x, y \in \mathcal{M}^{sa}$ with finite discrete spectra and $\|x - y\|_\infty \leqslant 2$ and $\|x - y\|_{p/2} \leqslant 2$. We prove that $\| |x|^\theta - |y|^\theta \|_{p/\theta} \leqslant C_{p,\theta}$ for some $C_{p,\theta} > 0$.

We denote by $x=\sum_{i\in I}x_ip_i$ and $y=\sum_{i\in J}y_jq_j$ the spectral decompositions of x and y, so that $p_i=1_{\{x_i\}}(x)$ and $q_i=1_{\{y_i\}}(y)$.

We will rely on the formula (4), decompose $|x|^\theta-|y|^\theta=\sum_{i,j=-1}^1 a_i(|x|^\theta-|y|^\theta)b_i$ where $a_{-1}=1_{]\infty,0[}(x)$, $a_0=1_{\{0\}}(x)$ and $a_1=1_{]0,\infty[}(x)$ and similarly for b_i.

We use dyadic decompositions. Set $I_k=[2^{-k-1},2^{-k}[$, $J_k=]0,2^{-k-1}[$ for $k\geqslant 0$ and $I_{-1}=[1,\infty[$, $J_{-1}=]0,1[$, $J_{-2}=]0,\infty[$. The definition is made so that the sets $I_k\times J_{k-1}$ and $J_k\times I_k$ are disjoint with union $\{(x,y)\in]0,\infty[^2\mid \max\{x,y\}\in I_k\}$. Hence we have a partition $]0,+\infty[^2=\cup_{k\geqslant -1}(I_k\times J_{k-1}\cup J_k\times I_k)$. Define accordingly the maps $T_k^1(z)=\sum_{x_i\in I_k,y_j\in J_{k-1}}p_i\frac{x_i^\theta-y_j^\theta}{x_i-y_j}zq_j$, $T_k^2(z)=\sum_{x_i\in J_k,y_j\in I_k}p_i\frac{x_i^\theta-y_j^\theta}{x_i-y_j}zq_j$. For any $0<a\leqslant 1$, Lemma 3.1 says that $\|T_k^1\|_{L_a(\mathcal{M})\to L_a(\mathcal{M})}$ is dominated by $\Big\|\Big(\frac{x_i^\theta-y_j^\theta}{x_i-y_j}\Big)_{x_i\in I_k,y_j\in J_{k-1}}\Big\|_{acb}$. By Lemma 2.4, this norm is smaller than $\Big\|\Big(\frac{x^\theta-y^\theta}{x-y}\Big)_{x\in\overline{I}_k,y\geqslant 0}\Big\|_{acb}$. Using Remark 2.10 and Corollary 2.9 for $k\geqslant 0$, this norm is bounded by $C_a2^{-k(\theta-1)}$. When $k=-1$, using Corollary 2.11 instead, we also get a bound by $C_{a,\theta}$. We have similar estimates for $\|T_k^2\|_{L_a(\mathcal{M})\to L_a(\mathcal{M})}$.

Put $r_k=1_{I_k}(x)$, $s_k=1_{I_k}(y)$ as well as $u_k=1_{J_k}(x)$, $v_k=1_{J_{k-1}}(y)$, we can write (note that the sums are actually finite)

$$\begin{aligned} a_1(|x|^\theta-|y|^\theta)b_1 &= \sum_{x_i>0,y_j>0}p_i\frac{x_i^\theta-y_j^\theta}{x_i-y_j}(x-y)q_j\\ &= \sum_{k\geqslant -1}T_k^1(x-y)+T_k^2(x-y)\\ &= \sum_{k\geqslant -1}T_k^1(r_k(x-y)v_k)+T_k^2(u_k(x-y)s_k).\end{aligned}$$

We use the above norm estimates for T_k^j with $a=p/\theta$ to get $\|T_k^j\|_{L_{p/\theta}(\mathcal{M})\to L_{p/\theta}(\mathcal{M})}\leqslant C_{p,\theta}2^{k(1-\theta)}$. By the p/θ-triangular inequality

$$\|a_1(|x|^\theta-|y|^\theta)b_1\|_{p/\theta}^{p/\theta}\leqslant C_{p,\theta}^{p/\theta}\sum_{k\geqslant -1}2^{k(1-\theta)p/\theta}(\|r_k(x-y)v_k\|_{p/\theta}^{p/\theta}+\|u_k(x-y)s_k\|_{p/\theta}^{p/\theta}).$$

But by definition $0\leqslant r_kx,u_kx,yv_k,ys_k\leqslant 2^{-k}$ for $k\geqslant 0$ and $\|x-y\|_\infty\leqslant 2$, so that we have $\|r_k(x-y)v_k\|_\infty\leqslant\|r_kx-yv_k\|_\infty\leqslant 2^{-k}$, and also $\|u_kx-ys_k\|_\infty\leqslant 2^{-k}$ for all $k\geqslant -1$. But also $\|r_k(x-y)v_k\|_{p/2},\|u_k(x-y)s_k\|_{p/2}\leqslant\|x-y\|_{p/2}\leqslant 2$. Thus as $\theta/p=(\theta/2).2/p+(1-\theta/2)/\infty$, by the Hölder inequality $\|r_k(x-y)v_k\|_{p/\theta},\|u_k(x-y)s_k\|_{p/\theta}\leqslant 2.2^{-k(1-\theta/2)}$. This is enough to conclude that

$$\|a_1(|x|^\theta-|y|^\theta)b_1\|_{p/\theta}^{p/\theta}\leqslant 4C_{p,\theta}^{p/\theta}\sum_{k\geqslant -1}2^{-kp/2}.$$

To deal with $a_1(|x|^\theta-|y|^\theta)b_{-1}$, one can do exactly the same using Corollary 2.14 as the Schur multipliers involved have shape $\Big(\frac{a^\theta-b^\theta}{a+b}\Big)_{a\in I_k,b\in J_{k-1}}$, $\Big(\frac{a^\theta-b^\theta}{a+b}\Big)_{a\in J_k,b\in I_k}$.

The terms $a_{-1}(|x|^\theta-|y|^\theta)b_1$ and $a_{-1}(|x|^\theta-|y|^\theta)b_{-1}$ can be treated similarly.

It is a well known fact that complex interpolation remains valid for $L_p(\mathcal{M},\tau)$ in the range $0<p\leqslant 1$, see Lemma 2.5 in [18] for what we need. Using it, one easily deals with the remaining terms as for instance

$$\|a_0(|x|^\theta-|y|^\theta)\|_{p/\theta}=\|a_0|y|^\theta\|_{p/\theta}\leqslant\|a_0|y|\|_p^\theta=\|a_0y\|_p^\theta=\|a_0(x-y)\|_p^\theta\leqslant 2.$$

Gluing the pieces together, we find a constant $C_{p,\theta}$ so that (5) holds in Case 1.

Case 2: We assume $\mathcal{M}$ finite, $x,y\in\mathcal{M}^{sa}$, $y=x+q$ for some projection q with $\|q\|_p=1$, that is $\tau(q)=1$.

Consider (x_n) a sequence in the von Neumann algebra generated by x that $|x_n|\leqslant|x|$, $\|x_n-x\|_\infty\to 0$ and each x_n has a finite spectrum, and similarly for (y_n) and y.

Obviously $\||x_n|^\theta-|x|^\theta\|_{p/\theta}\to 0$ by the dominated convergence theorem as $\mathcal{M}$ is finite (similarly for y). As $\|x_n-y_n\|_t\to\|q\|_t=1$ for $t=\infty,p/2$, x_n and y_n satisfy the assumptions of Case 1 for n big enough. Going to the limit in (5) for x_n,y_n, we get (5) for x,y with the same $C_{p,\theta}$.

Case 3: We assume $\mathcal{M}$ finite, $x, y \in \mathcal{M}^{sa}$ and $y = x + tq$ for some projection q with $\tau(q) = 1$ and $t \in \mathbb{R}$.

If $t > 0$, this follows by homogeneity applying the result for y/t, x/t and q. If $t < 0$, one just needs to exchange x and y. The constant $C_{p,\theta}$ is the same as in the previous cases.

Case 4: We assume $\mathcal{M}$ finite, $x, y \in \mathcal{M}^{sa}$ and $y = x + \sum_{i=1}^n t_i q_i$ where $t_i \in \mathbb{R}$ and q_i are orthogonal projections with $\tau(q_i) = 1$.

Simply put $x_0 = x$ and $x_k = x + \sum_{i=1}^k t_i q_i$ and use the p/θ-triangular inequality and Case 3 for x_{k-1} and x_k.

$$\|f(x) - f(y)\|_{p/\theta}^{p/\theta} \leqslant \sum_{i=1}^n \|f(x_{i-1}) - f(x_i)\|_{p/\theta}^{p/\theta} \leqslant C_{p,\theta}^{p/\theta} \sum_{i=1}^n \big(\|t_i q_i\|_p^\theta\big)^{p/\theta}.$$

But $\|x - y\|_p = \Big(\sum_{i=1}^n |t_i|^p\Big)^{1/p}$ and we get this case as $\|q_i\|_p = 1$.

Case 5: We assume $\mathcal{M}$ finite, $x, y \in \mathcal{M}^{sa}$ and $y = x + \sum_{i=1}^n t_i q_i$ where $t_i \in \mathbb{R}$ and q_i are orthogonal projections with $\tau(q_i) \in \mathbb{Q}$.

Consider the von Neumann algebra $(\tilde{\mathcal{M}}, \tilde{\tau})$ where $\tilde{\mathcal{M}} = \mathcal{M} \otimes L_\infty([0,1])$ with the trace $\tilde{\tau} = N\tau \otimes \int$ where N is such that for all i, $N\tau(q_i) \in \mathbb{N}$. Put $\tilde{z} = z \otimes 1$ for $z \in \mathcal{M}$. We have $\tilde{y} - \tilde{x} = \sum_{i=1}^n t_i \tilde{q}_i$. For each i, and $k \leqslant N\tau(q_i)$, let $q_{i,k} = q_i \otimes 1_{[(k-1)/(N\tau(q_i)), k/(N\tau(q_i))]}$. The $(q_{i,k})_k$ are orthogonal projections with $\tilde{\tau}(q_{i,k}) = 1$ and $\sum_{k=1}^{N\tau(q_i)} q_{i,k} = \tilde{q}_i$. By Case 4:

$$N^{\theta/p}\|f(x) - f(y)\|_{L_{p/\theta}(\mathcal{M})} = \|f(\tilde{x}) - f(\tilde{y})\|_{L_{p/\theta}(\tilde{\mathcal{M}})} \leqslant C_{p,\theta}\|\tilde{x} - \tilde{y}\|_{L_p(\tilde{\mathcal{M}})}^\theta = C_{p,\theta}\Big(N^{1/p}\|x - y\|_{L_p(\mathcal{M})}\Big)^\theta.$$

We still obtain the same constant $C_{p,\theta}$.

Case 6: We assume $\mathcal{M}$ finite and $x, y \in \mathcal{M}^{sa}$.

By considering again $(\tilde{\mathcal{M}}, \tilde{\tau})$ where $\tilde{\mathcal{M}} = \mathcal{M} \otimes L_\infty([0,1])$ with the trace $\tilde{\tau} = \tau \otimes \int$, in a masa containing it, one can approximate $x - y$ for the L_1-norm by elements of the form $\delta_k = \sum_{i=1}^{n_k} t_{i,k} q_{i,k}$ where $\tilde{\tau}(q_{i,k}) \in \mathbb{Q}$. With $y_k = x + \delta_k$, y_k converges to y in L_1 it also does in L_p as $\tilde{\mathcal{M}}$ is finite. The Ando inequality for the index $1/\theta$ also gives that $\|f(y_k) - f(y)\|_{1/\theta} \leqslant C_\theta \|y_k - y\|_1^\theta$, hence $f(y_k)$ converges to $f(y)$ in $L_{p/\theta}$. Since x, y_k satisfy the assumptions of Case 5, the conclusion follows by going to the limit in (5).

Case 7: We assume $\mathcal{M}$ semi-finite and $x, y \in L_p(\mathcal{M})^{sa}$.

This is again a matter of approximation. By the semi-finiteness of $\mathcal{M}$ and functional calculus in a masa containing x, we may approximate x in L_p by elements of the form $x_n = \sum_{i=1}^N t_i q_n$ commuting with x where q_n are finite projections. We can as well assume that $f(x_n)$ converges to $f(x)$ in $L_{p/\theta}$, similarly for y_n and y (these are purely commutative results). Then y_n and x_n are in some finite subalgebra of $\mathcal{M}$ and we have (5) for them by Case 6: $\|f(x_n) - f(y_n)\|_{p/\theta} \leqslant C_{p,\theta}\|x_n - y_n\|_p^\theta$. Taking again limits as $n \to \infty$ gives the result. □

Remark 3.3. One can get a proof of the case $p = \infty$ following case 1 (since $p/2 = \infty$) and then case 6 directly. A careful analysis of the constant shows that $\limsup_{\theta \to 1}(1 - \theta)C_{\infty,\theta} < \infty$ as in [3].

We slightly extend the result to τ-measurable operators $L_0(\mathcal{M}, \tau)$. With its measure topology [29, 14], it becomes a complete Hausdorff topological $*$-algebra (Theorem 28 in [29]) in which $\mathcal{M}$ is dense. We recall the Fatou Lemma in L_0 (Lemma 3.4 in [14]) if $v_n \to v$ in L_0 and then $\|v\|_q \leqslant \liminf \|v_n\|_q$ for all q. Moreover, the functional calculus associated to our $f(t) = |t|^\theta$ or $f(t) = \mathrm{sgn}(t)|t|^\theta$ is continuous on L_0^{sa} (Lemma 3.2 in [23]).

Theorem 3.4. *Let $0 < \theta < 1$ and $0 < p \leqslant \infty$ then there exists $C_{p,\theta}$ so that for any semi-finite von Neumann algebra $(\mathcal{M}, \tau)$, and $x, y \in L_0(\mathcal{M}, \tau)^{sa}$ such that $x - y \in L_p(\mathcal{M}, \tau)$, then $|x|^\theta - |y|^\theta, \mathrm{sgn}(x)|x|^\theta - \mathrm{sgn}(y)|y|^\theta \in L_{p/\theta}(\mathcal{M}, \tau)$ and*

$$\||x|^\theta - |y|^\theta\|_{p/\theta} \leqslant C_{p,\theta}\|x - y\|_p^\theta, \qquad \|\mathrm{sgn}(x)|x|^\theta - \mathrm{sgn}(y)|y|^\theta\|_{p/\theta} \leqslant C_{p,\theta}\|x - y\|_p^\theta.$$

Proof. This is a matter of approximations.

We start by giving arguments to get the result when $x, y \in \mathcal{M}^{sa}$ and $x - y \in L_p(\mathcal{M})$. First note that if a bounded sequence $(x_n)_n \in \mathcal{M}^{sa}$ goes to x for the strong-topology then $(f(x_n))_n$ also goes to $f(x)$ for the strong-topology (Lemma 4.6 in [28]). Take any finite projection $\gamma\in\mathcal{M}$ and $(q_n)_n$ a sequence of finite projections that goes to 1 strongly, it follows that $(\gamma f(q_nxq_n)\gamma)_n$ goes to $\gamma f(x)\gamma$ in L_2. since $L_2\subset L_0$ is continuous, we get that in the topology of L_0, $\lim_n \gamma f(q_nxq_n)\gamma=\gamma f(x)\gamma$ and similarly for y.

As q_nxq_n and q_nyq_n are in L_p, we get

$$\|\gamma(f(q_nxq_n)-f(q_nyq_n))\gamma\|_{p/\theta}\leqslant C_{p,\theta}\|q_n(x-y)q_n\|_p^\theta\leqslant C_{p,\theta}\|x-y\|_p^\theta.$$

Using the Fatou lemma, we obtain

$$\|\gamma(f(x)-f(y))\gamma\|_{p/\theta}\leqslant C_{p,\theta}\|x-y\|_p^\theta.$$

It is a simple exercise to check that if $z\in L_0^{sa}$ is so that $\sup\|\gamma z\gamma\|_{p/\theta}\leqslant C$ where the sup runs over all finite projections, then $z\in L_{p/\theta}$ with norm less than C. This allows to conclude.

Next take any $x,y\in L_0$ such that $x-y\in L_p$. Let $q_n=1_{|x|\leqslant n}\wedge 1_{|y|\leqslant n}$, then q_n is an increasing sequence of projections to 1 with $\tau(1-q_n)\to 0$ as $x,y\in L_0$. Moreover $q_nxq_n\in\mathcal{M}$ goes to x in L_0 (similarly for y) (see [29] page 20). By the continuity of the functional calculus in L_0, $(f(q_nxq_n))$ and $(f(q_nyq_n))$ go to $f(x)$ and $f(y)$. Once again with the help of the Fatou lemma and the previous case

$$\|f(x)-f(y)\|_{p/\theta}\leqslant\liminf\|f(q_nxq_n)-f(q_nyq_n)\|_{p/\theta}\leqslant C_{p,\theta}\liminf\|q_n(x-y)q_n\|_p^\theta\leqslant C_{p,\theta}\|x-y\|_p^\theta.$$

□

4. Ando's inequality in type III algebras

Before going to type III algebras, we need to extend Ando's inequality to weak-L_p spaces. We will use the K-interpolation method see [7].

As usual, given two compatible quasi-Banach spaces X_0 and X_1, we let for $t>0$ and $x\in X_0+X_1$

$$K_t(x,X_0,X_1)=\inf\{\|x_0\|_{X_0}+t\|x_1\|_{X_1}\ ;\ x=x_0+x_1\}.$$

For $0<\eta<1$ and $0<q\leqslant\infty$ set

$$\|x\|_{\eta,q}=\|t^{-\eta}K_t(x,X_0,X_1)\|_{L_q(\mathbb{R}^+,dt/t)}=\left(\int_0^\infty(t^{-\eta}K_t(x,X_0,X_1))^q\frac{dt}{t}\right)^{1/q},$$

with the obvious modification when $q=\infty$.

The interpolated space $(X_0,X_1)_{\eta,q}$ is $\{x\in X_0+X_1\mid \|x\|_{\eta,q}<\infty\}$ with (quasi)-norm $\|.\|_{\eta,q}$.

If $(\mathcal{M},\tau)$ is a semi-finite von Neumann algebra, and $x\in L_0(\mathcal{M},\tau)$ we denote as usual its decreasing rearrangement by $\mu_t(x)$ (see [14]).

The noncommutative Lorentz spaces $L_{p,q}(\mathcal{M},\tau)$ for $0<p<\infty$ and $0<q\leqslant\infty$ are defined as in the commutative case. The space $L_{p,q}(\mathcal{M},\tau)$ consists of all measurable operators $x\in L_0(\mathcal{M},\tau)$ so that $\|x\|_{L_{p,q}}=\|t^{1/p}\mu_t(f)\|_{L_q(\mathbb{R}^+,dt/t)}<\infty$. With the (quasi)-norm $\|.\|_{L_{p,q}}$, it becomes a (quasi)-Banach space.

The results about real interpolation of commutative L_p-spaces ([7] Theorem 5.3.1) remain available for semi-finite von Neumann algebras. Indeed for any $0<p_0<p_1\leqslant\infty$, for all $t>0$, $x\in L_{p_0}(\mathcal{M},\tau)+L_{p_1}(\mathcal{M},\tau)$, the quantities $K_t(x,L_{p_0}(\mathcal{M},\tau),L_{p_1}(\mathcal{M},\tau))$ and $K_t(\mu(x),L_{p_0}(\mathbb{R}^+),L_{p_1}(\mathbb{R}^+))$ are equivalent with constants depending only on p_0 and p_1, see [30] for details. As a consequence, we get that for $p_0\neq p_1\leqslant\infty$ and $0<q\leqslant\infty$, $0<\eta<1$ with $\frac{1}{p}=\frac{1-\eta}{p_0}+\frac{\eta}{p_1}$, $(L_{p_0}(\mathcal{M},\tau),L_{p_1}(\mathcal{M},\tau))_{\eta,q}=L_{p,q}(\mathcal{M},\tau)$ with equivalent norms (depending only on the parameters but not on $(\mathcal{M},\tau)$).

We drop the reference to $(\mathcal{M},\tau)$ to lighten notation.

When $x=x^*\in L_{p_0}+L_{p_1}$, we can also consider

$$K_t^{sa}(x,L_{p_0},L_{p_1})=\inf\{\|x_0\|_{p_0}+t\|x_1\|_{p_1}\ ;\ x=x_0+x_1\text{ with }x_i=x_i^*\}.$$

Lemma 4.1. *Let $0<p_0<p_1\leqslant\infty$, $t>0$ and $x\in(L_{p_0}+L_{p_1})^{sa}$, then*

$$K_t^{sa}(x,L_{p_0},L_{p_1})\geqslant K_t(x,L_{p_0},L_{p_1})\geqslant K_t^{sa}(x,L_{p_0},L_{p_1})/2^{\max\{1/p_0,1\}-1}.$$

Proof. This is a standard fact. Let $x=a_0+a_1$ with $a_i\in L_{p_i}$ such that $\|a_0\|_{p_0}+t\|a_1\|_{p_1}\leqslant K_t(x,L_{p_0},L_{p_1})+\epsilon$. Then $x=b_0+b_1$ with $b_i=(a_i+a_i^*)/2$. We have $\|b_i\|_{p_i}\leqslant 2^{1/p_i-1}\|a_i\|_{p_i}$ if $p_i<1$ or $\|b_i\|_{p_i}\leqslant\|a_i\|_{p_i}$ otherwise, hence we get the result letting $\varepsilon\to 0$. □

Remark 4.2. Similarly one can easily show that for $x \geqslant 0$, $K_t(x, L_{p_0}, L_{p_1})$ is equivalent to $\inf\{\|x_0\|_{p_0} + t\|x_1\|_{p_1} \ ; \ x = x_0 + x_1 \text{ with } x_i \geqslant 0\}$.

Lemma 4.3. *Let $0 < p_0 < p_1 \leqslant \infty$ and $\theta \in]0,1[$ and $x, y \in (L_{p_0} + L_{p_1})^{sa}$. Then for all $t > 0$ and $f(s) = |s|^\theta$ or $f(s) = \mathrm{sgn}(s)|s|^\theta$:*

$$K_{t^\theta}(f(y) - f(x), L_{p_0/\theta}, L_{p_1/\theta}) \leqslant C_{p_0,p_1,\theta} K_t(y - x, L_{p_0}, L_{p_1})^\theta.$$

Proof. Thanks to the previous Lemma, it suffices to do it with K_t^{sa} instead of K_t.

Choose selfadjoint operators $\delta_0 \in L_{p_0}$ and $\delta_1 \in L_{p_1}$ so that $y - x = \delta_0 + \delta_1$ and $\|\delta_0\|_{p_0} + t\|\delta_1\|_{p_1} \leqslant 2K_t^{sa}(x - y, L_{p_0}, L_{p_1})$.

Set $a_0 = f(y) - f(x + \delta_1) = f(y) - f(y - \delta_0)$ and $a_1 = f(x + \delta_1) - f(x)$, then $f(y) - f(x) = a_0 + a_1$. By Theorem 3.4 for p_i, we obtain $\|a_i\|_{p_i/\theta} \leqslant C_{p_i,\theta}\|\delta_i\|_{p_i}^\theta$. Since, $\|a_0\|_{p_0/\theta} + t^\theta\|a_1\|_{p_1/\theta} \leqslant C_{p_0,p_1,\theta}(\|\delta_0\|_1 + t\|\delta_1\|_{p_1})^\theta$, we have found a suitable decomposition to conclude. □

Proposition 4.4. *For all $0 < p < \infty$, $0 < q \leqslant \infty$ and $0 < \theta < 1$, there exists $C_{p,q,\theta} > 0$ such that for $x, y \in L_p^{sa}$, with $f(s) = |s|^\theta$ or $f(s) = \mathrm{sgn}(s)|s|^\theta$:*

$$\|f(y) - f(x)\|_{L_{p/\theta,q}} \leqslant C_{p,q,\theta}\|y - x\|_{L_{p,q\theta}}^\theta.$$

Proof. Put $p_0 = p/2$, $p_1 = 2p$ and $\eta = 2/3$ so that $\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$. We have

$$\|f(y) - f(x)\|_{L_{p/\theta,q}} \simeq_{p/\theta,q} \|t^{-\eta}K_t(f(y) - f(x), L_{p_0/\theta}, L_{p_1/\theta})\|_{L_q(\mathbb{R}^+,dt/t)},$$
$$\|y - x\|_{L_{p,q\theta}} \simeq_{p,q\theta} \|u^{-\eta}K_u(y - x, L_{p_0}, L_{p_1})\|_{L_{q\theta}(\mathbb{R}^+,du/u)}.$$

But by Lemma 4.3

$$\|t^{-\eta}K_t(f(y) - f(x), L_{p_0/\theta}, L_{p_1/\theta})\|_{L_q(\mathbb{R}^+,dt/t)} \leqslant C_{p,\theta}\|t^{-\eta}K_{t^{1/\theta}}(y - x, L_{p_0}, L_{p_1})^\theta\|_{L_q(\mathbb{R}^+,dt/t)}.$$

But by a change of variable $u = t^{1/\theta}$:

$$\|t^{-\eta}K_{t^{1/\theta}}(y - x, L_{p_0}, L_{p_1})^\theta\|_{L_q(\mathbb{R}^+,dt/t)} = \theta^{1/q}\|u^{-\eta}K_u(y - x, L_{p_0}, L_{p_1})\|_{L_{q\theta}(\mathbb{R}^+,du/u)}^\theta$$

and we get the estimate. □

We can conclude with the proof of Theorem 3.2 for type III algebras.

Proof. Assume that $\mathcal{M}$ is given with a n.s.f. weight φ with modular group σ. Let $\mathcal{R} = \mathcal{M} \rtimes_{\hat\sigma} \mathbb{R}$ be its core (with the dual action $\hat\sigma$), this is a semi-finite von Neumann algebra with a trace τ such that $\tau \circ \hat\sigma_t = e^{-t}\tau$. Then by definition $L_p(\mathcal{M}, \varphi)$ is isometrically a subspace of $L_{p,\infty}(\mathcal{R}, \tau)$, more precisely

$$L_p(\mathcal{M}, \varphi) = \{x \in L_0(\mathcal{R}, \tau) \mid \hat\sigma_t(x) = e^{-t/p}x, \ \forall t \in \mathbb{R}\} \subset L_{p,\infty}(\mathcal{R}, \tau),$$

with by definition $\|x\|_p = \|x\|_{L_{p,\infty}(\mathcal{R})}$ (see [29] Definition 13 and Lemma 5 or Lemma B in [16]).

Thus Theorem 3.2 for type III algebras follows from Proposition 4.4 for the weak-L_p spaces noticing that $f(L_p(\mathcal{M}, \varphi)) \subset L_{p/\theta}(\mathcal{M}, \varphi)$ as $\hat\sigma_t$ is a representation and f is θ-homogeneous. □

5. Further comments

By very classical arguments using the Cayley transform see [24], estimates for the functional calculus are equivalent to some for commutators or anticommutators.

Proposition 5.1. *Let $0 < p \leqslant \infty$ and $0 < \theta < 1$, then there exists a constant $C_{p,\theta}$ so that for $x \in L_p(\mathcal{M})^{sa}$ and $b \in \mathcal{M}$ with $f(s) = |s|^\theta$ or $f(s) = \mathrm{sgn}(s)|s|^\theta$:*

$$\big\|[f(x), b]\big\|_{p/\theta} \leqslant C_{p,\theta}\big\|[x, b]\big\|_p^\theta \|b\|^{1-\theta}.$$

If $x, y \in L_p(\mathcal{M})^+$ and $b \in \mathcal{M}$, then

$$\big\|bx^\theta \pm y^\theta b\big\|_{p/\theta} \leqslant C_{p,\theta}\big\|bx \pm yb\big\|_p^\theta \|b\|^{1-\theta}.$$

Let $M_{p,q}$ denote the Mazur map for $p < q$ given by $M_{p,q}(f) = f|f|^{(p-q)/q}$. The 2×2-tricks from [24] also give

Proposition 5.2. *Let $0 < p \leqslant \infty$ and $0 < \theta < 1$, then there exists a constant $C_{p,\theta}$ so that for $x, y \in L_p(\mathcal{M})$:*

$$\big\|M_{p,p/\theta}(x) - M_{p,p/\theta}(y)]\big\|_{p/\theta} \leqslant C_{p,\theta}\|x - y\|_p^\theta.$$

This improves the estimates for $M_{p,q}$ when $p < q$ of Theorem 4.1 in [18] and gives the expected Hölder continuity.

Remark 5.3. The above two propositions are valid for all von Neumann algebras. For semi-finite ones, similar estimates are true for the Lorentz norms as in Proposition 4.4.

Contrary to the case $p \geqslant 1$, at least for $p \leqslant 1/2$, there is no constant C_p so that for all $0 < \theta < 1$, $\|\Big(\frac{x^\theta - y^\theta}{x-y}\Big)_{x\geqslant 1, y\geqslant 1}\|_{pcb} \leqslant C_p$.

Indeed if this is so, then taking $\theta = 1-\varepsilon$, $x = e^{t/\varepsilon}$, $y = e^{s/\varepsilon}$ by Lemma 2.5, letting $\varepsilon \to 0$, we would get that $\|\Big(e^{-\max\{t,s\}}\Big)_{t,s\geqslant 0}\|_{pcb} \leqslant C_p$. Using that $2\max\{x,y\} = |x-y| + x + y$, we would deduce that $\|\Big(e^{-|t-s|}\Big)_{0\leqslant t,s\leqslant 1}\|_{pcb} \leqslant C_p$.

By easy arguments going from a discrete situation to a continuous one, one can deduce that the same matrix has to be a Schur multiplier on $S^p(L_2[0,1])$. The constant kernel 1 on $[0,1]^2$ is in $S^p(L_2[0,1])$ with norm one, thus we would get that $\|\Big(e^{-|x-y|}\Big)_{0\leqslant x,y\leqslant 1}\|_{S^p(L_2[0,1])} \leqslant C_p$.

The operator T on $L_2[0,1]$ with kernel $K(x,y) = e^{-|x-y|}$ is obviously positive and Hilbert-Schmidt with HS norm less than 1. One easily checks that the eigenvectors f_λ associated to λ must satisfy $\lambda f'' = \lambda f - 2f$. Letting $\lambda = \frac{2}{1+\alpha^2}$ with $\alpha > 0$, the only possibilities are $f_\lambda(x) = ae^{\mathrm{i}\alpha x} + be^{-\mathrm{i}\alpha x}$. But $T(e^{\mathrm{i}\alpha x}) = \frac{2}{1+\alpha^2}e^{\mathrm{i}\alpha x} - \frac{e^{-x}}{1+\mathrm{i}\alpha} - e^{x}\frac{e^{-1+\mathrm{i}\alpha}}{1-\mathrm{i}\alpha}$. Thus, λ is a eigenvalue iff $e^{2\mathrm{i}\alpha} = \Big(\frac{1-\mathrm{i}\alpha}{1+\mathrm{i}\alpha}\Big)^2$. Set $\tan\theta = \alpha$ with $\theta \in]0,\pi/2[$, so that $e^{2\mathrm{i}\alpha} = e^{-4\mathrm{i}\theta}$. The equation $\tan t = -2t + k\pi$ admits a unique solution θ_k in $]0,\pi/2[$ for $k > 0$ so that $\tan\theta_k \approx k\pi$. We deduce that the set of eigenvalues of T is $(\frac{2}{1+\tan(\theta_k)^2})_{k\geqslant 1}$ with associated eigenvector $f_\lambda(x) = e^{\mathrm{i}\alpha x} - \frac{1+\mathrm{i}\alpha}{1-\mathrm{i}\alpha}e^{-\mathrm{i}\alpha x}$. Thus $T \notin S^p(L_2[0,1])$ when $p \leqslant 1/2$.

As $f_\lambda/\|f_\lambda\|_2$ is uniformly bounded in $\mathcal{C}[0,1]$, we can also deduce using Lemma 2.3 that for $p > 1/2$

$$\|\Big(e^{-\max\{t,s\}}\Big)_{0\leqslant t,s\leqslant 1}\|_{pcb} < \infty \quad \text{and} \quad \|\Big(e^{-\max\{t,s\}}\Big)_{t,s\geqslant 0}\|_{pcb} < \infty.$$

About Corollary 2.11, it is easier to see that the norm cannot be independent of $0 < \theta < 1$ for $p \leqslant 1$. Indeed the multiplier $\Big(\frac{x-y}{x+y}\Big)_{x\geqslant 1, y\geqslant 1}$, or equivalently $\Big(\frac{x-y}{x+y}\Big)_{x\geqslant 0, y\geqslant 0}$ using homogeneity, is not bounded for $p = 1$ (hence also for $p < 1$) as shown in [12].

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