Helmholtz Equation in Highly Heterogeneous Media: a two Scales Analysis

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Heterogeneous Helmholtz equation

- The pressure $u$ satisfies the equation

$$-\frac{\omega^2}{c^2} u - \Delta u = f$$

- Where $f$ is a source of excitation
- Where $\omega$ is the angular frequency
- Where $c$ is the wave velocity
Heterogeneous Helmholtz equation

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Frequency

- the frequency is represented by the parameter $\omega \in \mathbb{R}^*_+$
- solving for high frequencies requires heavy computations
- high order methods may reduce the computational cost
- coarse mesh with high order elements
Heterogeneous Helmholtz equation

- The pressure $u$ satisfies the equation

\[- \frac{\omega^2}{c^2} u - \Delta u = f\]

- Where $f$ is a source of excitation
- Where $\omega$ is the angular frequency
- Where $c$ is the wave velocity

Wave velocity

- The wave velocity is represented as a function $c : \Omega \rightarrow \mathbb{R}^*_+$
- In geophysical applications, $c$ is piecewise constant
- In classical FEM, the mesh has to fit $c$
- $c$ must be constant in each cell of the mesh
- For highly heterogenous media, we need a fine mesh
Existence and Uniqueness: Homogeneous case

- Fredholm Alternative
- Uniqueness (scaling argument)
- Existence
- Stability ($C(\omega)$)

Frequency explicit stability

- Use a special test function in the variational formulation
- Frequency explicit bounds:

$$|u|_{0,\Omega} \leq \frac{C}{\omega}|f|_{0,\Omega}, \quad |u|_{1,\Omega} \leq C|f|_{0,\Omega}, \quad |u|_{2,\Omega} \leq C\omega|f|_{0,\Omega},$$

with $C = C(\Omega)$
Sketch of the proof: Homogeneous case

- Varational form

\[ B(u, v) = -\frac{\omega^2}{c^2} \int_\Omega u \bar{v} - i\frac{\omega}{c} \int_{\partial\Omega} u \bar{v} \int_\Omega \nabla u \cdot \nabla \bar{v} \]

- Test function \( v = \nabla u \cdot x \)

\[ 2\text{Re} \left(-\frac{\omega^2}{c^2} \int_\Omega u \bar{v}\right) = \frac{\omega^2}{c^2} \int_\Omega |u|^2 - \frac{\omega^2}{c^2} \int_{\partial\Omega} |u|^2 x \cdot n \]
**Sketch of the proof: Heterogeneous case**

- **Varational form**

\[
B(u, v) = -\omega^2 \int_{\Omega} \frac{1}{c^2} u \bar{v} - i\omega \int_{\partial\Omega} \frac{1}{c} u \bar{v} \int_{\Omega} \nabla u \cdot \nabla \bar{v}
\]

- **Test function** \( v = \nabla u \cdot x \)

\[
2\text{Re} - \omega^2 \int_{\Omega} \frac{1}{c^2} u \bar{v} = \omega^2 \int_{\Omega} \frac{1}{c^2} |u|^2 - \omega^2 \int_{\partial\Omega} \frac{1}{c^2} |u|^2 x \cdot n
\]

\[
- \sum_{r,l} \int_{\Omega_r \cap \Omega_l} \left( \frac{1}{c^2_r} x \cdot n_r + \frac{1}{c^2_l} x \cdot n_l \right) |u|^2
\]
Sketch of the proof: Heterogeneous case

- **Variational form**

\[
B(u, v) = -\omega^2 \int_{\Omega} \frac{1}{c^2} u \bar{v} - i \omega \int_{\partial\Omega} \frac{1}{c} u \bar{v} \int_{\Omega} \nabla u \cdot \nabla \bar{v}
\]

- **Test function** \( v = \nabla u \cdot x \)

\[
2 \Re \left( -\omega^2 \int_{\Omega} \frac{1}{c^2} u \bar{v} \right) = -\omega^2 \int_{\Omega} \frac{1}{c^2} |u|^2 - \omega^2 \int_{\partial\Omega} \frac{1}{c^2} |u|^2 x \cdot n
\]

\[
- \sum_{r,l} \int_{\Omega_r \cap \Omega_l} \left( \frac{1}{c_r^2} x \cdot n_r + \frac{1}{c_l^2} x \cdot n_l \right) |u|^2
\]

**Additional Hypothesis**

\[
\frac{1}{c_r^2} x \cdot n_r + \frac{1}{c_l^2} x \cdot n_l \leq 0, \quad \forall x \in \Omega_r \cap \Omega_l, \quad \forall r, l
\]
A Stratified medium

\[
(0, 0) \quad (L_1, 0) \\
(0, L_2) \quad + x_0
\]
A Stratified medium

(0, 0) (L1, 0)

(0, L2)

x0

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Helmholtz Equation in Highly Heterogeneous media
Helmholtz Equation
Analytical study
Numerical analysis
Numerical experiments

Homogeneous case
Heterogeneous case

A Stratified medium

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Helmholtz Equation in Highly Heterogeneous media
A Stratified medium

The point \((0,0)\) lies on the left boundary of the domain, \((0,L_2)\) lies on the bottom edge, and \((L_1,0)\) lies on the right boundary.

The blue line represents the path \(x_0\) through the medium, indicating the propagation of waves or other phenomena across the stratified layers.
A Stratified medium

(0, 0) (L1, 0)
(0, L2) x0

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A Stratified medium

The trajectory represented in the diagram is defined by the point $(0, 0)$ and $(L_1, 0)$. The medium is stratified with different layers indicated by the contour lines for the values 2000, 2200, 2500, 2800, 3000, and 3500. The point $x_0$ is marked on the trajectory.
A Stratified medium

\[(0, 0) \quad \text{to} \quad (L_1, 0)\]

\[(0, L_2) \quad \text{to} \quad x_0\]
A Salt body

Helmholtz Equation in Highly Heterogeneous media

Homogeneous case
Numerical experiments
Numerical analysis
Analytical study

Heterogeneous case
Helmholtz Equation
Analytical study
Numerical analysis
Numerical experiments

Homogeneous case
Heterogeneous case

Stability (Heterogeneous case)

- Additional hypothesis on $c$
- Same bounds than the homogeneous case
- $C = C(\Omega, \frac{c_{\text{max}}}{c_{\text{min}}})$
Helmholtz Equation

Settings

Homogeneous case

Numerical experiments

Analytical study

Numerical analysis

Numerical experiments

Settings

- $\mathcal{T}_H$ mesh of $\Omega$
- $\mathcal{T}_H$ conforming, regular
- $H$ mesh step
- $V_H^p$ discretisation space of order $p$

$$V_H^p = \{ v \in C^0(\bar{\Omega}) \mid v|_K \in P_p(K) \quad \forall K \in \mathcal{T}_H \}$$
Homogeneous case

- Stability is proved under the condition that

\[ \omega^{p+1} H^p \leq C, \]

- then we have

\[ \omega |u - u_H|_{0,\Omega} + |u - u_H|_{1,\Omega} \leq C \omega^{p+2} H^{p+1}, \]

for all \( \omega > 0 \).
Addional Constraint

- The parameter $c$ must be taken into account

A solution

- Fix $H$ and $p$ to fit $\omega$
- Use subquadrature schemes of step $h$
- Fix $h$ to fit $c$
Numerical Integration

- One must integrate quantities
  \[ \int_K \frac{1}{c^2} \varphi^i \varphi^j, \quad \varphi^i, \varphi^j \in \mathcal{P}_p(K), \quad K \in \mathcal{T}_H \]

- Consider a submesh \( \mathcal{T}_{H,h}^K \) of \( K \)
- Replace \( c \) by \( c_h \)
- \( c_h \) piecewise constant on \( \mathcal{T}_{H,h}^K \)

\[
\int_K \frac{1}{c^2} \varphi^i \varphi^j \simeq \int_K \frac{1}{c_h^2} \varphi^i \varphi^j = \sum_{B \in \mathcal{T}_{H,h}^K} \frac{1}{c_h^2} \int_B \varphi^i \varphi^j
\]
Numerical Integration

- Consider a mesh $\hat{T}_h$ of $\hat{K}$
- Compute the reference integrals

$$\hat{i}^i,j_B = \int_{\hat{B}} \phi^i \phi^j, \quad \phi^i, \phi^j \in P_p(\hat{K}), \quad \hat{B} \in \hat{T}_h$$

- Map them using the Jacobian

$$\int_K \frac{1}{c^2 h} \phi^i \phi^j = \text{jac} J_K \sum_{\hat{B} \in \hat{T}_h} \frac{1}{c^2 h} \hat{i}^i,j_B$$
Helmholtz Equation
Analytical study
Numerical analysis
Numerical experiments

Settings
Homogeneous case
Heterogeneous case

Mapping

$A_c$
$\hat{K}$
$F_K$
$K$

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**Stability and convergence**

- Linear elements $p = 1$
- Stability is ensured if

$$\omega^2 H + \omega h \leq C,$$

and then

$$\omega |u - u_H|_{0,\Omega} + |u - u_H|_{1,\Omega} \leq C (\omega h + \omega^3 H^2)$$
Two-layer medium \((\omega = 10\pi, c_1 = 1000, c_2 = 2000, \mathcal{P}_4 \text{ elements})\)
Two-layer medium \((\omega = 10\pi, \, c_1 = 1000, \, c_2 = 2000, \, P_4 \text{ elements})\)

Reference cell
Two-layer medium \((\omega = 10\pi, c_1 = 1000, c_2 = 2000, \mathcal{P}_4 \text{ elements})\)

Reference cell

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Helmholtz Equation in Highly Heterogeneous media
## Multi-layer test case

- 1000 layers of 3 meters each
- $c_{\text{min}} = 500 \text{ m/s}$, $c_{\text{max}} = 5500 \text{ m/s}$
- $|c_j - c_{j+1}| \geq 1000$
- $\omega = 40\pi$ ($f = 20 \text{ Hz}$)
- $P_6$ elements
Multi-layer test case

- 1000 layers of 3 meters each
- $c_{\text{min}} = 500 \text{m.s}^{-1}$, $c_{\text{max}} = 5500 \text{m.s}^{-1}$
- $|c_j - c_{j+1}| \geq 1000$
- $\omega = 40\pi$ ($f = 20$ Hz)
- $P_6$ elements

Mesh

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Helmholtz Equation in Highly Heterogeneous media
Without subquadrature (7.44 relative $L^2$ error)
Helmholtz Equation
Analytical study
Numerical analysis
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Two-layer medium
Multi-layer medium

64 subcells ($4.45 \times 10^{-1}$ relative $L^2$ error)
1024 subcells ($7.00 \times 10^{-2}$ relative $L^2$ error)
Conclusion

- subquadrature schemes capture fine scale heterogeneities
- an arbitrary high order 2D solver has been implemented
- theoretical convergence issues have been addressed
- but the estimates are not sharp
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- an arbitrary high order 2D solver has been implemented
- theoretical convergence issues have been addressed
- but the estimates are not sharp

Perspective

- a 3D solver is on the way
- sharper estimates
- non-constant density

\[- \frac{\omega^2}{\kappa} u - \text{div} \left( \frac{1}{\rho} \nabla u \right) = f\]